KTH Mathematics

SF2812 Applied linear optimization, final exam
Thursday October 18 2012 14.00–19.00

Examiner: Anders Forsgren, tel. 08-790 71 27.

Allowed tools: Pen/pencil, ruler and eraser. *Note!* Calculator is not allowed.

Solution methods: Unless otherwise stated in the text, the problems should be solved by systematic methods, which do not become unrealistic for large problems. Motivate your conclusions carefully. If you use methods other than what have been taught in the course, you must explain carefully.

Note! Personal number must be written on the title page. Write only one exercise per sheet. Number the pages and write your name on each page.

22 points are sufficient for a passing grade. For 20-21 points, a completion to a passing grade may be made within three weeks from the date when the results of the exam are announced.

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1. Consider the linear programming problem (LP) defined as

$$(LP) \quad \begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b, \\ & x \geq 0, \end{array}$$

where

$$A = \begin{pmatrix} 2 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 1 & 2 & 0 & 0 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 5 \\ 1 \\ 7 \end{pmatrix},$$

$$c = \begin{pmatrix} -1 & 1 & 1 & 0 & 0 \end{pmatrix}^T.$$

A friend of yours claims that she has computed an optimal solution $\widehat{x} = (3\ 2\ 3\ 1\ 0)^T$. However, she is a bit confused since she would expect an optimal solution to have at most three positive variables.

Help your friend by providing two basic feasible solutions with the same objective function value as $\widehat{x}$. Start from $\widehat{x}$. Finally, verify optimality of one of these basic feasible solutions. ....(10p)

Hint: Your may find one or several of the results below useful.

$$\begin{pmatrix} 2 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 1 & 2 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ 0 \\ -2 \\ -3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} 2 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 1 & 2 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} 2 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 1 & 2 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ 3 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

2. Consider the linear program

$$(LP) \qquad \begin{array}{ll} \min & c^T x \\ \text{s.t.} & Ax = b, \\ & x \geq 0, \end{array}$$

where

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 4 \\ 4 \end{pmatrix}, \quad c = \begin{pmatrix} 0 & 1 & 2 & 1 \end{pmatrix}^T.$$

Assume that we want to solve (LP) using a primal-dual interior-point method. Assume further that we initially choose $x^{(0)} = (1\,2\,3\,4)^T$, $y^{(0)} = (0\,0)^T$, $s^{(0)} = (4\,3\,2\,1)^T$. Here, y and s denote the dual variables.

(a) Formulate the system of linear equations to be solved in the first iteration of the primal-dual interior-point method for the given initial values. First formulate the general form and then add explicit numerical values into the system of equations. Select an appropriate value of the barrier parameter. ....(7p)

(b) Assume that the system of linear equations has been solved, giving a solution $\Delta x^{(0)}$, $\Delta y^{(0)}$, $\Delta s^{(0)}$. Assuming that $\Delta x^{(0)}$, $\Delta y^{(0)}$ and $\Delta s^{(0)}$ are known, explain how $x^{(1)}$, $y^{(1)}$ and $s^{(1)}$ would be determined. ....(3p)

3. Consider the stochastic program (P) given by

$$(P) \qquad \begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b, \\ & T(\omega)x = h(\omega), \\ & x \geq 0, \end{array}$$

where ω is a stochastic variable and $T(\omega)x = h(\omega)$ is to be interpreted as an "informal" stochastic constraint. Assume that ω takes on a finite number of values $\omega_1, \dots, \omega_N$ with corresponding probabilities $p_1, \dots, p_N$. Let T_i denote $T(\omega_i)$ and let h_i denote $h(\omega_i)$.

(a) Explain how the deterministically equivalent problem

$$\begin{array}{ll} \text{minimize} & c^T x + \sum_{i=1}^{N} p_i q_i^T y_i \\ \text{subject to} & Ax = b, \\ & T_i x + W y_i = h_i, \quad i = 1, \dots, N, \\ & x \geq 0, \\ & y_i \geq 0, \quad i = 1, \dots, N, \end{array}$$

arises. (We assume, for simplicity, "fix compensation", i.e., W does not depend on i.) .... (6p)

(b) Define *VSS* in terms of suitable optimization problems. .... (2p)

(c) Define *EVPI* in terms of suitable optimization problems. .... (2p)

4. Consider the linear program (LP) given by

$$(LP) \quad \begin{array}{ll} \text{minimize} & 2x_1 - 2x_2 + 3x_3 \\ \text{subject to} & x_1 + 4x_2 - 3x_3 = 0 \\ & -1 \leq x_j \leq 1, \quad j = 1, 2, 3. \end{array}$$

Your task is to solve (LP) using Dantzig-Wolfe decomposition taking into account problem structure.

Treat the equality constraint $x_1 + 4x_2 - 3x_3 = 0$ as the hard constraint. For $S = \{x \in I\!R^3 : -1 \leq x_j \leq 1, j = 1, 2, 3\}$, write $x \in S$ as a convex combination of the extreme points of S. In the master problem, start with the basis that corresponds to the extreme points $(1 \ \ 1 \ \ 1)^T$ and $(-1 \ -1 \ -1)^T$. The subproblem(s) that arise(s) may be solved by inspection. .... (10p)

5. Consider the integer program (IP) given by

$$(IP) \quad \begin{array}{ll} \text{minimize} & 2x_1 - 2x_2 + 3x_3 \\ \text{subject to} & x_1 + 4x_2 - 3x_3 = 0 \\ & x_j \in \{-1, 0, 1\}, \quad j = 1, 2, 3. \end{array}$$

Associated with (IP) we may define the dual problem (D) as

$$(D) \quad \begin{array}{ll} \text{maximize} & \varphi(u) \\ \text{subject to} & u \in I\!R, \end{array}$$

where $\varphi(u) = \min\{2x_1 - 2x_2 + 3x_3 - u(x_1 + 4x_2 - 3x_3) : x_j \in \{-1, 0, 1\}, j = 1, 2, 3\}$.

(a) Solve (D) by first determining $\varphi(u)$ for $u \in I\!R$ explicitly, and then using this expression for $\varphi(u)$ to find an optimal solution u^*. You may solve the Lagrangian relaxation problem(s) that gives $\varphi(u)$ in any way, that need not be systematic. .... (5p)

(b) At the optimal solution u^*, determine two subgradients to φ derived from optimal solutions to the corresponding Lagrangian relaxation problem. Again, you need not use a systematic method for generating the optimal solutions to the Lagrangian relaxation problem. .... (3p)

(c) What can you say about the relationship between the optimal value of (D) and the optimal value of (LP) of question 4? Motivate the answer. .... (2p)

Good luck!