SF2812 Applied linear optimization, final exam
Wednesday June 8 2016 8.00–13.00

Examiner: Anders Forsgren, tel. 08-790 71 27.

Allowed tools: Pen/pencil, ruler and eraser. *Note!* Calculator is not allowed.

Solution methods: Unless otherwise stated in the text, the problems should be solved by systematic methods, which do not become unrealistic for large problems. Motivate your conclusions carefully. If you use methods other than what have been taught in the course, you must explain carefully.

Note! Personal number must be written on the title page. Write only one exercise per sheet. Number the pages and write your name on each page.

22 points are sufficient for a passing grade. For 20-21 points, a completion to a passing grade may be made within three weeks from the date when the results of the exam are announced.

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1. Consider a transportation problem (TP) defined as

$$(TP)\quad \begin{array}{ll} \text{minimize} & \displaystyle\sum_{i=1}^{2}\sum_{j=1}^{3} c_{ij}x_{ij} \\ \text{subject to} & \displaystyle\sum_{j=1}^{3} x_{ij} = a_i, \quad i = 1,2, \\ & \displaystyle\sum_{i=1}^{2} x_{ij} = b_j, \quad j = 1,2,3, \\ & x_{ij} \geq 0, \quad i = 1,2, \ j = 1,2,3, \end{array}$$

where

$$C = \begin{pmatrix} 2 & 0 & 1 \\ 0 & -3 & -2 \end{pmatrix}, \quad a = \begin{pmatrix} 8 \\ 5 \end{pmatrix}, \quad b = \begin{pmatrix} 6 \\ 5 \\ 2 \end{pmatrix}.$$

The dual problem associated with (TP) may be written as

$$(DTP)\quad \begin{array}{ll} \text{maximize} & \displaystyle\sum_{i=1}^{2} a_i u_i + \sum_{j=1}^{3} b_j v_j \\ \text{subject to} & u_i + v_j \leq c_{ij}, \quad i = 1,2, \quad j = 1,2,3. \end{array}$$

You have been given $\widehat{X}$, $\widehat{u}$ and $\widehat{v}$ as

$$\widehat{X} = \begin{pmatrix} 6 & 1.5 & 0.5 \\ 0 & 3.5 & 1.5 \end{pmatrix}, \quad \widehat{u} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \quad \widehat{v} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$$

(a) A friend of yours, who has not taken this course, claims that $\widehat{X}$ cannot be optimal to (TP), since the transportation problem should have integer valued optimal solutions when a and b are integers. Comment on your friend's claim. .... (2p)

(b) Verify that $\widehat{X}$ is optimal to (TP) and that $\widehat{u}$, $\widehat{v}$ is optimal to (DTP). ... (3p)

(c) Find, using $\widehat{X}$, two integer valued optimal solutions to (TP). .... (3p)
Hint: It holds that $\sum_{i=1}^{2}\sum_{j=1}^{3} c_{ij}w_{ij} = 0$, $\sum_{j=1}^{3} w_{ij} = 0$, $i = 1, 2$, and $\sum_{i=1}^{2} w_{ij} = 0$, $j = 1, 2, 3$, for

$$W = \begin{pmatrix} 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix}.$$

(d) Explain why you would not obtain $\widehat{X}$ as an answer if you used the simplex method to solve (TP). .... (2p)

2. Let $S = \{x : Ax = b, x \geq 0\}$. Assume that S has extreme points v_i, $i = 1, \ldots, k$. Show that S may be written as

$$\begin{aligned} S &= \left\{ x : x = d + \sum_{i=1}^{k} v_i \alpha_i,\ Ad = 0,\ d \geq 0,\ \sum_{i=1}^{k} \alpha_i = 1,\ \alpha \geq 0 \right\} \\ &= \left\{ x : x = d + V\alpha,\ Ad = 0,\ d \geq 0,\ e^T\alpha = 1,\ \alpha \geq 0 \right\}, \end{aligned}$$

where $V = \begin{pmatrix} v_1 & v_2 & \cdots & v_k \end{pmatrix}$ and $e = \begin{pmatrix} 1 & 1 & \cdots & 1 \end{pmatrix}^T$. .... (10p)

3. Consider the linear program

$$(LP) \qquad \begin{array}{ll} \min & c^Tx \\ \text{s.t.} & Ax = b, \\ & x \geq 0, \end{array}$$

where

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 4 \\ 4 \end{pmatrix}, \quad c = \begin{pmatrix} 0 & 1 & 2 & 1 \end{pmatrix}^T.$$

Assume that we want to solve (LP) using a primal-dual interior-point method. Assume further that we initially choose $x^{(0)} = (4\,3\,2\,1)^T$, $y^{(0)} = (0\,0)^T$, $s^{(0)} = (1\,2\,3\,4)^T$. Here, y and s denote the dual variables.

(a) Formulate the system of linear equations to be solved in the first iteration of the primal-dual interior-point method for the given initial values. First formulate the general form and then add explicit numerical values into the system of equations. Select an appropriate value of the barrier parameter. ....(7p)

(b) Assume that the system of linear equations has been solved, giving a solution $\Delta x^{(0)}$, $\Delta y^{(0)}$, $\Delta s^{(0)}$. Assuming that $\Delta x^{(0)}$, $\Delta y^{(0)}$ and $\Delta s^{(0)}$ are known, explain how $x^{(1)}$, $y^{(1)}$ and $s^{(1)}$ would be determined. .... (3p)

4. Consider the binary integer programming problem (IP) given by

$$(IP) \qquad \begin{array}{ll} \text{minimize} & -5x_1 - 7x_2 - 10x_3 \\ \text{subject to} & -3x_1 - 6x_2 - 7x_3 \geq -8, \\ & -x_1 - 2x_2 - 3x_3 \geq -3, \\ & x_j \in \{0,1\}, \quad j = 1, \ldots, n. \end{array}$$

Consider the dual problem obtained when the constraint $-3x_1 - 6x_2 - 7x_3 \geq -8$ is relaxed by Lagrangian relaxation for a nonnegative multiplier u.

(a) Illustrate the dual problem graphically. Give the optimal solution and the optimal value to the dual problem. You may use any method to solve the problems that arise, that need not be systematic. . . . . (7p)

(b) Your not so reliable friend AF claims that the optimal value to the linear programming relaxation problem corresponding to (IP) is –10. Based on your solution to Question 4a, explain to AF why he is wrong. . . . . (3p)

5. Consider a cutting-stock problem with the following data:

$$W = 12, \quad m = 3, \quad w_1 = 3, \quad w_2 = 5, \quad w_3 = 9, \quad b = \begin{pmatrix} 60 & 50 & 50 \end{pmatrix}^T.$$

Notation and problem statement are in accordance to the textbook. Given are rolls of width W. Rolls of m different widths are demanded, where roll i has width w_i, $i = 1, \ldots, m$. The demand for roll i is given by b_i, $i = 1, \ldots, m$. The aim is to cut the W-rolls so that a minimum number of W-rolls are used.

(a) Solve the the LP-relaxed problem associated with the above problem. Start with the basic feasible solution associated with the three "pure" cut patterns $(4\ 0\ 0)^T$, $(0\ 2\ 0)^T$ and $(0\ 0\ 1)^T$. The subproblems that arise may be solved in any way, that need not be systematic. . . . . (8p)

(b) Determine a "near-optimal" solution to the original problem. Give a bound on the maximum deviation from the optimal value of the original problem. . . (2p)

Good luck!