KTH Mathematics

SF2822 Applied nonlinear optimization, final exam
Saturday June 2 2012 9.00–14.00
Brief solutions

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1. As $g_2(x^*) > 0$ we must have $g_2(x) \geq 0$.

Since $g_1(x^*) = 0$, $g_3(x^*) = 0$, with $\nabla g_1(x^*)$ and $\nabla g_3(x^*)$ linearly independent, it follows that x^* is a regular point. Hence, the first-order necessary optimality conditions must hold. We therefore try to find λ_1 and λ_3 such that

$$\begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \lambda_1 + \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \lambda_3.$$

There is a solution for $\lambda_1 = -1$ and $\lambda_3 = 2$. Since $\lambda_1 < 0$ and $\lambda_3 > 0$, we must have $g_1(x) \leq 0$ and $g_3(x) \geq 0$ for the first-order necessary optimality conditions to hold.

We now investigate whether this choice gives a local minimizer. The Jacobian of the active constraints at x^* is given by

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}.$$

As the first two columns form an invertible matrix, we may for example obtain Z from

$$Z = \begin{pmatrix} -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} -1 \\ -1 \end{pmatrix} \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

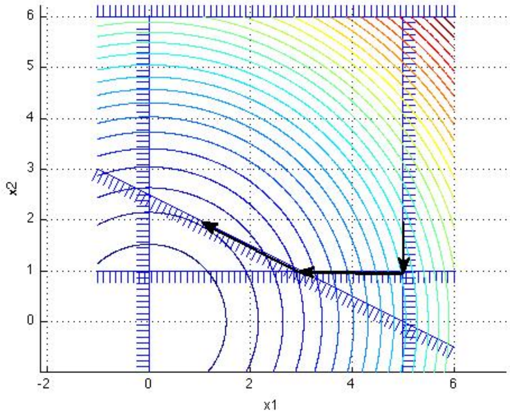
Hence,

$$\begin{aligned} Z^T(\nabla^2 f(x^*) - \lambda_1 \nabla^2 g_1(x^*) - \lambda_3 \nabla^2 g_3(x^*))Z &= \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \\ &= -1, \end{aligned}$$

which is not a positive semidefinite matrix. Therefore, x^* is a regular point at which the second-order necessary optimality conditions do not hold. Consequently, x^* is not a local minimizer.

We conclude that it is not possible to replace "?" by "$\leq$" or "$\geq$" so that x^* becomes a local minimizer to (NLP).

2. The iterations are illustrated in the figure below.



In the first iteration, the search direction points towards the minimizer with constraint 3 active, $(5\ 0)^T$, but the step is limited by constraint 2 at $(5\ 1)^T$. Hence, the next iterate is $(5\ 1)^T$ and constraint 2 is added.

In the second iteration, the search direction is zero so that the multipliers are evaluated. We have $\lambda_2 = 1$, $\lambda_3 = -5$. Constraint 3 has a negative multiplier, so that the next iterate remains $(5\ 1)^T$ and constraint 3 is deleted.

In the third iteration, the search direction points towards the minimizer with constraint 2 active, $(0\ 1)^T$, but the step is limited by constraint 5 at $(3\ 1)^T$. Hence, the next iterate is $(3\ 1)^T$ and constraint 5 is added.

In the fourth iteration, the search direction is zero so that the multipliers are evaluated. We have $\lambda_2 = -5$, $\lambda_5 = 3$. Constraint 2 has a negative multiplier, so that the next iterate remains $(3\ 1)^T$ and constraint 2 is deleted.

In the fifth iteration, the search direction points towards the minimizer with constraint 5 active, $(1\ 2)^T$, which is feasible. The multiplier of constraint 5 is positive, $\lambda_5 = 1$, so the optimal solution has been found.

3. We have

$$f(x) = 4(x_1-2)^2 + (x_2-1)^2 \qquad g(x) = 1 - x_1^2 - x_2^2 \geq 0,$$

$$\nabla f(x) = \begin{pmatrix} 8(x_1-2) \\ 2(x_2-1) \end{pmatrix}, \qquad \nabla g(x) = \begin{pmatrix} -2x_1 \\ -2x_2 \end{pmatrix},$$

$$\nabla^2 f(x) = \begin{pmatrix} 8 & 0 \\ 0 & 2 \end{pmatrix}, \qquad \nabla^2 g(x) = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}.$$

The first QP-subproblem becomes

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}p^T\nabla^2_{xx}\mathcal{L}(x^{(0)},\lambda^{(0)})p + \nabla f(x^{(0)})^T p \\ \text{subject to} & \nabla g(x^{(0)})^T p \geq -g(x^{(0)}, \end{array}$$

Insertion of numerical values gives

$$\begin{array}{ll} \text{minimize} & 4p_1^2 + p_2^2 \\ \text{subject to} & -4p_1 - 2p_2 \geq 4. \end{array}$$

We now utilize the fact that the problem is of dimension two with only one constraint. The constraint must be active, since the unconstrained minimizer $p = 0$ is infeasible. Hence, we may let $p_1 = -1 - p_2/2$ and minimize

$$4(1 + \frac{p_2}{2})^2 + p_2^2.$$

Setting the derivative to zero gives

$$0 = 4(1 + \frac{p_2}{2}) + 2p_2 = 4 + 4p_2.$$

Hence, $p_2 = -1$, which gives $p_1 = -1/2$. Evaluating the gradient at the optimal point of the quadratic program gives

$$\begin{pmatrix} -4 \\ -2 \end{pmatrix} = \begin{pmatrix} -4 \\ -2 \end{pmatrix} \lambda,$$

so that $\lambda = 1$. Consequently, we obtain

$$x^{(1)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} -\frac{1}{2} \\ -1 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} \\ 0 \end{pmatrix}, \quad \lambda^{(1)} = 1.$$

4. (See the course material.)

5. (a) The Hessian of the objective function is given by

$$\nabla^2 f(x) = \begin{pmatrix} \frac{2x_2^2}{x_1^3} & -\frac{2x_2}{x_1^2} \\ -\frac{2x_2}{x_1^2} & \frac{2}{x_1} \end{pmatrix}$$

If we let $A = 2/x_1$, $B = -2x_2/x_1^2$ and $C = 2x_2^2/x_1^3$ we obtain $A > 0$ and $C - B^2/A = 0$ if $x_1 > 0$. Hence, Hint 2 shows that $\nabla^2 f(x) \succeq 0$ for $x_1 > 0$. Thus, f is convex on the convex feasible region, so that (NLP) is a convex problem.

(b) Problem (NLP) is equivalent to (NLP') given by

$$(NLP') \qquad \begin{array}{ll} \text{minimize} & x_3 \\ \text{subject to} & x_3 \geq \frac{x_2^2}{x_1}, \\ & x \in F, \end{array}$$

since $x_3 = x_2^2/x_1$ must hold at any optimal solution. For $x_1 > 0$ we obtain

$$x_3 \geq \frac{x_2^2}{x_1} \quad \Leftrightarrow x_3 - \frac{x_2^2}{x_1} \geq 0 \quad \Leftrightarrow \begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix} \succeq 0,$$

where Hint 2 has been used in the last step with $A = x_1$, $B = x_2$ and $C = x_3$. Hence, (NLP') is equivalent to the semidefinite program

$$\begin{array}{ll} \text{minimize} & x_3 \\ \text{subject to} & \begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix} \succeq 0, \\ & x \in F. \end{array}$$