KTH Matematik

Solution to Homework 1
Mathematical Systems Theory, SF2832
Fall 2008
You may use min(3,(your score)/10) as bonus credit on the exam.

1. Solve the following linear state equations

(*a*) $\dot{x}(t) = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix} x(t),\ x(0) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ .... (2p)

$$\mathbf{e^{At}} = \begin{bmatrix} e^{-t} & e^{-t} - e^{-2t} & 0 \\ 0 & e^{-2t} & 0 \\ 0 & 0 & e^{3t} \end{bmatrix}.$$

(*b*) $\dot{x}(t) = (1 + sin(t))x(t),\ \ x(0) = 2.$ .... (2p)

$$\mathbf{x(t) = 2e^{t-\cos(t)+1}}.$$

(*c*) Let an $n \times n$ matrix $A(t)$ be continuous and diagonal. Show the state transition matrix $\Phi(t, s) = exp(\int_s^t A(r)dr)$. .... (2p)

Proof: In this case $A(t)$ and $\int_s^t A(r)dr$ commute.

2. Consider the system

$$\dot{x}(t) = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u(t),$$

where $\lambda_1, \lambda_2, \lambda_3$ are real numbers.

(*a*) Determine when the system is controllable (reachable) .... (5p)

When λ_i are distinctive.

(*b*) Let $\lambda_1 = 2,\ \lambda_2 = \lambda_3 = 1$. Make a change of variables for which the system has the decomposed representation

$$\begin{bmatrix} \dot{x}_c(t) \\ \dot{x}_{\bar{c}}(t) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} x_c(t) \\ x_{\bar{c}}(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u(t)$$

.... (5p)

Let for example $\bar{x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} x$.

3. In this problem you will investigate whether it is possible to reconstruct the initial condition from measurements of the output for the system

$$\dot{x}(t) = Ax(t),\ x(0) = x_0$$
$$y(t) = Cx(t)$$

where

$$A = \begin{bmatrix} 0 & 1 \\ -1 & a \end{bmatrix},$$

where a is a constant.

(a) Suppose you can measure $x_1(t)$, namely $C = [1\ 0]$, for what a is it possible to reconstruct the initial state x_0? .... (2p)
For all a.

(b) Suppose you can measure $x_2(t)$, namely $C = [0\ 1]$, for what a is it possible to reconstruct the initial state x_0? .... (3p)
For all a.

(c) Suppose $a = 0$ and $C = [0\ 1]$. If you can measure $y(t)$ at $t = 0, \pi/2, \pi, 3\pi/2$, is it possible to reconstruct the initial state x_0? .... (3p)
$x(t) = (\cos(t)x_{10} + \sin(t)x_{20}, -\sin(t)x_{10} + \cos(t)x_{20})^T$. $y(t) = -\sin(t)x_{10} + \cos(t)x_{20}$. **Then we only need to verify**

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \\ 0 & -1 \\ 1 & 0 \end{bmatrix}$$

has rank 2.

4. Let $\Phi(t,s) = \begin{bmatrix} \Phi_{11}(t,s) & \Phi_{12}(t,s) \\ \Phi_{21}(t,s) & \Phi_{22}(t,s) \end{bmatrix}$ be the state transition matrix for $A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$

(a) Show that $\Phi_{21}(t,s) = 0\ \forall t,\ s$. .... (3p)
Since $\Phi(s,s) = I$ and $\frac{\partial}{\partial t}\Phi_{21}(t,s) = A_{22}\Phi_{21}(t,s)$ with $\Phi_{21}(s,s) = 0$, thus $\Phi_{21}(t,s) = 0$.

(b) Show that $\frac{\partial}{\partial t}\Phi_{ii}(t,s) = A_{ii}\Phi_{ii}(t,s),\ i = 1,2$. .... (4p)
It follows from a).

5. (a) Consider the harmonic oscillator

$$\dot{x} = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} x, \quad x(0) = x_0$$

where the angular velocity ω is a real scalar. For any $x_0 \neq 0$, show the trajectory $\{x(t) : t \geq 0\}$ forms a circle. What is the radius of the circle? .... (3p)
Since $\frac{d}{dt}|x(t)|^2 = 0,\ |x(t)|^2 = |x_0|^2$.

(b) Consider the dynamics for a rotation matrix $R(t)$:

$$\dot{R}(t) = \Omega R, \qquad R(0) = R_0$$

where, $\Omega = \begin{bmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{bmatrix}$, and the angular velocities ω_1, ω_2, and ω_3 are real scalars, and $R_0^T R_0 = I$. Show that for any $t > 0$, $R^T(t)R(t) = I$. .... (4p)

Proof: Since $\Omega^T = -\Omega$ (skew symmetric), thus $R^T(t)R(t) = R_0^T R_0$.