KTH Matematik

Solution to Homework 2
Mathematical Systems Theory, SF2832
Fall 2008
You may use min(3,(your score)/10) as bonus credit on the exam.

1. Consider the pair (C, A), where

$$A = \begin{bmatrix} 0 & 1 \\ a_1 & a_2 \end{bmatrix}$$
$$C = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

For what a_1 and a_2 the Lyapunov equation $A^T P + PA + C^T C = 0$ has a positive definite solution? .... (6p)

Answer: $a_1 < 0, \quad a_2 < 0$.

2. You will in this problem derive and investigate a number of realizations for the transfer function

$$R(s) = \begin{bmatrix} \dfrac{\gamma}{s+1} & \dfrac{s+2}{s+1} \\ \dfrac{1}{s+1} & \dfrac{s+2}{s+1} \end{bmatrix},$$

where $\gamma > 0$ is a constant.

(a) Determine the standard reachable realization of $R(s)$. .... (4p)
Answer: Straight forward calculation.

(b) Is the realization in (a) observable? .... (2p)
Answer: yes if $\gamma \neq 1$.

(c) Determine the standard observable realization of $R(s)$. .... (4p)
Answer: Straight forward calculation.

3. Suppose the following is a realization of a given $r(s)$:

$$(A, B, C) = \left(\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix} \right)$$

(a) Is this realization minimal? .... (3p)
Answer: no.

(b) If not, use Kalman decomposition to find a minimal realization. . . . . (5p)
Answer: the key here is to find an appropriate transformation matrix P.
Since

$$\begin{aligned} R^4 &= \mathcal{R} \cap ker\Omega + V_{or} + V_{\bar{o}\bar{r}} + V_{\bar{r}o} \\ &= Im \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} + Im \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} + \{0\} + Im \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \end{aligned}$$

we can obtain P accordingly. This results in

$$\dot{x} = x + u, \quad y = x.$$

4. Consider

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx, \end{aligned}$$

where,

$$(A, B, C) = \left(\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 & 3 & 1 \end{bmatrix} \right)$$

(a) Is $x = 0$ asymptotically stable when u is set to zero? . . . . (2p)
Answer: no, since 0 is an eigenvalue.

(b) Is the controlled system BIBO-stable when the initial state is set to zero? (2p)
Answer: yes.

(b) Discuss if A being a stable matrix is necessary for BIBO stability. . . . . (3p)
Answer: no, as is shown by this example.

5. Consider a time-invariant controllable system

$$\dot{x} = Ax + Bu.$$

Let $W(T) = \int_0^T e^{-A\tau} BB^T e^{-A^T\tau} d\tau$, where T is an arbitrary positive number.
Show that if $u = -B^T W^{-1}(T)x$, then the overall system is asymptotically stable (Hint: consider Lyapunov function $x^T W^{-1} x$). . . . . (8p)

Proof: the key in the proof is to use the following fact:
Let $K(t) = e^{-At} BB^T e^{-A^T t}$ and $\bar{A} = A - BB^T W^{-1}(T)$, then

$$\dot{K} = -AK - KA^T.$$

Thus $e^{-AT} BB^T e^{-A^T T} - BB^T = -AW(T) - W(T)A^T$, which implies

$$\bar{A}W(T) + W(T)\bar{A}^T = -BB^T - e^{-AT} BB^T e^{-A^T T} \leq -BB^T.$$

Using Corollary 4.1.11 in the compendium we can draw the conclusion.