KTH Matematik

Solution to Homework 3
Mathematical Systems Theory, SF2832
Spring 2012
For reference only.

1. Consider

$$\dot{x} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & n \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix} x.$$

(a) Is the system controllable? .... (1p)
Solution: Yes.

(b) Can a stabilizing feedback controller $u = kx$ (i.e. the closed-loop poles are all placed at the open left half plane) make the system unobservable? .... (2p)
Solution: The original system is obviously also observable. To answer the question is to check if there are zeros of the transfer function on the left half plane. Since $r(s) = \sum_{i=1}^{n} \frac{1}{s-i} = \sum_{i=1}^{n} \frac{\lambda - i - j\sigma}{(\lambda-i)^2+\sigma^2}$, no zero can be with negative λ.

(d) Assume $n = 2$ and the full state is not available. Design an observer-based control that stabilizes the overall system, with the closed-loop poles located at $\{-1, -2\}$ and the observer dynamics having poles at $\{-1, -2\}$. .... (2p)

2. Consider

$$\dot{x} = Ax + Bu$$

where (A, B) is controllable and A does not have any eigenvalue on the imaginary axis. Given the cost function

$$J = \int_0^\infty u^T u dt,$$

we can show that $u = -B^T P x$ is the optimal control, where P is a positive *semi-definite* solution to the corresponding ARE.

(a) What are the eigenvalues of $(A - BB^T P)$? .... (2p)
Solution: First decompose A into the stable part and the anti-stable part, then apply the results in the solution to HW3 2011.

(b) What will happen to the optimal control problem if A has eigenvalues on the imaginary axis? (It is enough to use an as simple as possible example to explain). (3p)
Solution: Consider $\dot{x} = u$ and let $u = -kx$, then we can try to optimize k directly from the cost function ($u = -ke^{-kt}x_0$) and draw conclusion.

3. Consider

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= u,\end{aligned}$$

and the cost function

$$J = \int_0^{t_1} ((x_1 + x_2)^2 + u^2)dt.$$

(*a*) Find the optimal control. .... (3p)

(*b*) Let $t_1 = \infty$. What is the optimal control? .... (2p)

4. (*a*) Suppose (A, B) is controllable and (C, A) is observable, and P is the positive definite solution to

$$A^TP + PA - PBB^TP + C^TC = 0.$$

Show $A - kBB^TP$ is a stable matrix for all $k \geq 1$. .... (3p)

Solution: The ARE can be rewritten as

$$(A - kBB^TP)^T + P(A - kBB^TP) = -(2k - 1)PBB^TP - CC^T.$$

This shows that $A - kBB^TP$ is at least critically stable when $k \geq 1$ ($k > 0,5$ is enough). We then apply the same argument in p. 77 of the compendium to show that $A - kBB^TP$ is a stable matrix.

(*b*) Consider

$$\begin{aligned}x(t+1) &= \begin{bmatrix}1 & a\\ 0 & 1\end{bmatrix} x(t) + \begin{bmatrix}\frac{a^2}{2}\\ a\end{bmatrix} v(t)\\ y(t) &= \begin{bmatrix}1 & 0\end{bmatrix} x(t) + w(t),\end{aligned}$$

where v, w are uncorrelated white noises, with covariances $q,\ r$ respectively. Design Kalman filter for the system. .... (2p)