KTH Matematik

Solution to Homework 2
Mathematical Systems Theory, SF2832
Fall 2014
For reference only

1. Consider the pair (C, A), where

$$A = \begin{bmatrix} a_1 & a_2 \\ 1 & 0 \end{bmatrix}$$
$$C = \begin{bmatrix} 0 & 1 \end{bmatrix}.$$

For what a_1 and a_2 the Lyapunov equation $A^T P + PA + C^T C = 0$ has

(*a*) a positive definite solution? .... (1p)
Solution: Since (C, A) is observable, $P > 0$ iff A is a stable matrix, i.e. $a_1 < 0, a_2 < 0$.

(*b*) a negative definite solution (-P is positive definite)? .... (1p)
Solution: $a_1 > 0, a_2 > 0$

(*c*) a solution that is neither positive nor negative definite? Where are the eigenvalues of A located in this case? .... (2p)
Solution: When $a_1 a_2 \leq 0$. The two eigenvalues do not lie on the same open half of the complex plane.

2. You will in this problem derive and investigate a number of realizations for the transfer function

$$R(s) = \begin{bmatrix} \dfrac{\gamma}{s+1} & \dfrac{1}{s+1} \\ \dfrac{1}{(s+1)^2} & \dfrac{1}{(s+1)^2} \end{bmatrix},$$

where $\gamma > 0$ is a constant.

(*a*) Determine the standard reachable realization of $R(s)$. .... (1p)
Solution: $\chi(s) = s^2 + 2s + 1$, $N(s) = \begin{bmatrix} \gamma & 1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} \gamma & 1 \\ 0 & 0 \end{bmatrix} s$, the rest is omitted.

(*b*) What is the McMillan degree of $R(s)$? .... (2p)
Solution: $\delta(R) = 3$ if $\gamma \neq 1$, otherwise 2.

(*c*) Find a minimal realization of $R(s)$ for $\gamma = 1$. .... (3p)
Solution: In this case $\delta(R) = 2$. Since the first column of R equals the second, we just need to find a minimum realization for $\bar{R}(s) = \begin{bmatrix} \dfrac{1}{s+1} & \dfrac{1}{(s+1)^2} \end{bmatrix}^T$ and then replace u by $u_1 + u_2$. The standard controllable realization of $\bar{R}$ has dimension 2.

3. (*a*) Given the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_1 & -a_2 & -a_3 & \cdots & -a_n \end{bmatrix},$$

Suppose all the eigenvalues of A are real. Show that A is diagonalizable by a linear transformation T if and only if all the eigenvalues of A are distinct. (3p)

Solution: "if": straight forward. "only if": Suppose A is diagonalizable but not all eigenvalues are distinct, then there is no $b \in R^n$ that makes the diagonalized A and b controllable, but $b = [0 \cdots 0\, 1]^T$ makes (A, b) controllable, contradiction.

(*b*) Consider a controllable and observable system

$$\dot{x} = Ax + bu$$
$$y = cx,$$

where, $x \in R^n$, $u \in R$, $y \in R$. We say the system has relative degree r if $cb = 0, \cdots, cA^{r-2}b = 0$, and $cA^{r-1}b \neq 0$. Show we can find $u = kx = \sum_{i=1}^{n} k_i x_i$ such that $(c, A + bk)$ is not observable if and only if $r < n$. .... (3p)

Solution: Since the system is minimal, there is no zero-pole cancellation in $R(s) = c(sI - A)^{-1}b$. To introduce unobservability is to introduce zero-pole cancellation by feedback control, since this implies the feedback system is not minimal any more, meanwhile the controllability does not change. There are zeros to be canceled iff $r < n$.

(*c*) Consider

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = u,$$

one can use high gain control $u = -2k^2 x_1 - 3kx_2,\ k > 0$ to place the poles to $-k,\ -2k$. To see a drawback of having too high gain, show that for the closed-loop system if $|x_1(0)| \neq 0$

$$\lim_{k\to\infty} \max_{t\geq 0} |x(t)| = \infty.$$

.... (3p)

Solution: An easy way to see this is to choose $x(0) = (1, 0)^T$ and compute $e^{At}x(0)$.

4. Consider the inverted pendulum

The following equation describes the motion of the pendulum around the equilibrium $\theta = 0$:

If $|\theta(t)| < \frac{\pi}{2}$:

$$L\ddot{\theta} - g\sin(\theta) + \ddot{x}\cos(\theta) = 0, \tag{1}$$

If $|\theta(t)| = \frac{\pi}{2}$:

$$\dot{\theta} = 0,\ \ddot{\theta} = 0. \tag{2}$$

(2) indicates that once the pendulum falls on the cart, it remains in that position. Assume $L = 1$, and let $x_1 = \theta,\ x_2 = \dot{\theta},\ u = \ddot{x}$, we can linearize (1) as

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= gx_1 - u \end{aligned} \tag{3}$$

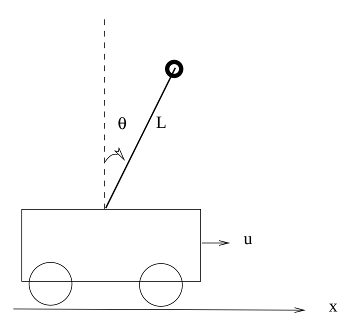
(*a*) Design $u = gx_1 + k_1x_1 + k_2x_2$, where $k_1 + k_2 \leq 30$, such that

- (3) is asymptotically stable, and
- use the nonlinear model (1) AND (2) and Matlab to find out what is the maximum $|\theta(0)|$ while $\dot{\theta}(0) = 0$ you can swing up. Give this maximum value with an accuracy of ± 0.1 degree and attach the simulation plot as evidence. .... (3p)

Note: If the best value is obtained by not more than two students, then the winner(s) will receive **two extra bonus credits** for the exam, provided that the winning control is allowed to be published on the course web.

(*b*) Now let $y = x_1$. Design an observer based on (3) and repeat the simulation. Attach a plot to show what is the maximum angle now. .... (3p)

Solution: We will announce the winning solution later.



Figur 1: Inverted pendulum on a cart.