KTH Matematik

Solution to Homework 3
Mathematical Systems Theory, SF2832
Fall 2014
For reference only.

1. Consider

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = u$$
$$x(0) = x_0.$$

In problem 3c. of the second homework, we have shown that high gain control can lead to unbounded x as the gain tends to infinity. Now let us try a different idea with high gain control.

(a) Design a stabilizing feedback control $u = k_1 x + k_2 x_2$ that is also the optimal control to

$$\min_u \int_0^\infty (x_1^2(t) + \frac{1}{h^2} u^2(t))dt,$$

here we can view h as the control gain. .... (3p)

Solution: to obtain the ARE we have $Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $R = \frac{1}{h^2}$. After solving ARE we have $u = -h^2 B^T P x = -(hx_1 + \sqrt{2h}x_2)$.

(b) What is the optimal cost $V(0)$ as $h \to \infty$? .... (3p)

Solution: As is expected, $V(0) = x_0^T P x_0 = 0$ as $h \to \infty$.

(c) Now let $y = x_1$ and design an observer gain $(l_1\ l_2)^T$ such that the observer functions also as optimal filter in steady state (as $t \to \infty$) if we consider u as noise of covariance $1\delta(t-s)$, and there is also a measurement noise w of covariance $\eta^2 \delta(t-s)$. .... (3p)

Solution: Let P be the solution to $AP + PA^T - PC^T R^{-1} CP + BB^T = 0$, where $R = \eta^2$. Let $P = -R\bar{P}^{-1}$, then $\bar{P}$ can be solved from the ARE in (a).

2. Let

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(a) Can we find $b \in R^4$ such that (A, b) is controllable? .... (1p)

Solution: Let $\bar{x}_3 = x_3 - x_4$, then one has converted A into the Jordan form

$$\bar{A} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and can see easily A is not cyclic. Thus no $b \in R^4$ can make (A, b) controllable.

(b) Find a 4×2 matrix B with as few non-zero elements as possible such that (A, B) is controllable. .... (3p)

Solution: We can easily find two controls, for example, $\bar{B} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ * & * & * & 1 \end{bmatrix}^T$ that makes $(\bar{A}, \bar{B})$ controllable. We can derive that we need 3 non-zero elements.

(c) Suppose $c = (c_1, c_2, c_3, c_4)$, together with A and B you found in (b) is a realization of some transfer matrix $R(s)$. What is the highest MacMillan degree $R(s)$ can have among all real elements of c? .... (2p)

Solution: Due to the duality between controllability and observability, Ω has rank maximum 3 since the largest cyclic sub-block in $\bar{A}$ is 3×3. Thus max. $\delta(R) = 3$.

3. Consider a controllable system

$$\dot{x} = Ax + Bu.$$

In the compendium we derive first the optimal control for a fixed end-point problem as

$$u = -B^T P(t)x(t) + B^T \Phi(t_1, t)^T W(t_0, t_1)^{-1}[x_1 - \Phi(t_1, t_0)x_0], \tag{1}$$

then use Bellman's principle to argue that the optimal control can be rewritten as

$$u = -B^T P(t)x(t) + B^T \Phi(t_1, t)^T W(t, t_1)^{-1}[x_1 - \Phi(t_1, t)x(t)]. \tag{2}$$

In this exercise, you are asked to show that (2) is equivalent to (1) without using Bellman's principle. .... (5p)

Solution: We plug in the first controller, then

$$\dot{x} = \bar{A}(t)x + BB^T \Phi(t_1, t)W(t_0, t_1)^{-1}[x_1 - \Phi(t_1, t_0)x_0],$$

where $\bar{A} = A - BB^T P(t)$, $\dot{\Phi} = \bar{A}\Phi$, $\Phi(t_0, t_0) = I$, $W(t_0, t_1) = \int_{t_0}^{t_1} \Phi(t_1, s)BB^T \Phi^T(t_1, s)ds$. Then,

$$\begin{aligned} x_1 &= \Phi(t_1, t)x(t) + \int_t^{t_1} \Phi(t_1, s)BB^T \Phi^T(t_1, s)W^{-1}(t_0, t_1)[x_1 - \Phi(t_1, t_0)x_0]ds \\ &= \Phi(t_1, t)x(t) + W(t, t_1)W^{-1}(t_0, t_1)[x_1 - \Phi(t_1, t_0)x_0] \end{aligned}$$

Then, $W^{-1}(t, t_1)[x_1 - \phi(t_1, t)x(t)] = W^{-1}(t_0, t_1)[x_1 - \Phi(t_1, t_0)x_0]$

Thus (1) is equivalent to (2).

4. At time $t = 1, 2, 3, \cdots$, an observation $y(t)$ is made of an unknown constant x. The observation error $y(t) - x$ is zero mean white noise with variance σ^2. Our apriori knowledge on x has variance p_0.

(a) Design a Kalman filter for the estimation of x. .... (1p)
Solution: The system model is

$$
\begin{aligned}
x(t+1) &= x(t) \\
y(t) &= x(t) + w(t),
\end{aligned}
$$

where $E\{w(t)w(s)\} = \sigma^2\delta(t,s),\ E\{x^2(1)\} = p_0$ (Note that the measurement arrives from time 1). The rest is omitted.

(b) Express the covariance matrix $p(t) = E\{(x - \hat{x}(t))^2\}$ in terms of $t,\ \sigma,\ p_0$. (2p)
Solution: $p(t+1) = \frac{1}{t\sigma^{-2}+p_0^{-1}}$.

(c) What is $\hat{x}(t)$ (expressed in terms of $y(1), \cdots, y(t-1)$) if we do not have any apriori knowledge on x? (Hint: what is p_0 in this case?) .... (2p)
Solution: In this case $p_0 = \infty$. If we let $\hat{x}(1) = 0$, then $\hat{x}(t+1) = \frac{1}{t}\sum_{k=1}^{t} y(k)$.