KTH Matematik

Solution to Homework 2
Mathematical Systems Theory, SF2832
Fall 2016
You may use min(5,(your score)/4) as bonus credit on the exam.

1. Consider a time-invariant system

$$\dot{x} = Ax,$$

where $x \in R^n$, $A \neq 0$ and $tr(A) = 0$, $tr(\cdot)$ denotes the trace of a matrix.

Show

(a) The system is not asymptotically stable (around $x = 0$). .... (2p)
Answer: Since $\sum \lambda_i = tr(A) = 0$, not all eigenvalues can have negative real parts.

(b) The system is not even (critically) stable if A is also symmetric. ....(1p)
Answer: In this case A is diagonalizable and all the eigenvalues are real, thus not all eigenvalues can be non-positive unless $A = 0$.

(c) The system is (critically) stable if A is also skew symmetric ($A^T = -A$). . (1p)
Answer: $\|x(t)\| = \|x(0)\|$.

2. Given the following system

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_1 & -a_2 & -a_3 & \cdots & -a_n \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} c_1 & c_2 \cdots & c_n \end{bmatrix} x.$$

(a) We say (C, A) is detectable if $Ce^{At}x_0 = 0,\ \forall t \geq 0$ implies $\lim_{t\to\infty} e^{At}x_0 = 0$. For the case $c_2 = 1,\ c_i = 0,\ i \geq 3$, discuss conditions on c_1 such that the system being detectable but not observable is possible. .... (3p)
Answer: (C, A) is detectable if and only if all canceled poles by the zeros are with negative real parts. This can be seen using Kalman decomposition (after possible pole-zero cancellation, we would have a minimal realization, and all minimal realizations are similar). This implies that $c_1 > 0$.

(b) Use Kalman decomposition to show that Theorem 4.3.4 in the compendium can be modified as: Assume (C, A) is detectable. Then A is a stable matrix iff $A^T P + PA + C^T C = 0$ has a positive semi-definite solution P such that $x^T Px > 0\ \forall x \notin ker\ \Omega$. .... (3p)
Answer: We can decompose $R^n = \mathcal{R} \cap ker\ \Omega + V_{or}$ (Note $\mathcal{R} = R^n$). This gives $P = \begin{bmatrix} 0 & 0 \\ 0 & P_2 \end{bmatrix}$, where P_2 is positive definite.

3. (a) Consider the system in 2.(a) and let $n = 3$, $c_3 = 1$. Find the feedback control $u = kx$ that makes $y(t) = 0\ \forall t \geq 0$ (this implies that we assume $y(0) = 0$). (2p)

Answer: $y(t) = 0$ implies $\dot{y}(t) = 0$ from which we can solve for u.

(b) Show that for all solutions $x(t)$ of the closed-loop system in 3.(a) such that $cx(t) = 0\ \forall t \geq 0$, $\lim_{t\to\infty} x(t) = 0$ iff $c_1 > 0$ and $c_2 > 0$. .... (2p)

Answer: When $y(t) = 0$, i.e. $x_3(t) = -c_1x_1(t) - c_2x_2(t)$, we have $\dot{x}_1 = x_2,\ \dot{x}_2 = -c_1x_1(t) - c_2x_2(t)$ and the conclusion follows.

4. Consider

$$R(s) = \begin{bmatrix} \dfrac{k}{s+1} & \dfrac{1}{s+2} \\[2ex] \dfrac{1}{s+1} & \dfrac{1}{s+1} \end{bmatrix},$$

where k is a constant.

(a) Determine the standard reachable realization of $R(s)$. .... (2p)

Answer: omitted.

(b) Determine the standard observable realization of $R(s)$. .... (2p)

Answer: omitted.

(c) What is the McMillan degree of $R(s)$? .... (2p)

Answer: $R(s) = 3$ if $k \neq 0$, $R(s) = 2$ if $k = 0$.