KTH Matematik

**Homework 1**
**Mathematical Systems Theory, SF2832**
**Fall 2017**
**You may use min(5,(your score)/4) as bonus credit on the exam.**

**1.** Solve the following linear state equations

$(a)$ $\dot{x}(t) = \begin{bmatrix} -1 & 1 & e^t \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} x(t),\ x(0) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . (2p)

$(b)$ $\dot{x}(t) = \sin(t)x(t) - \sin(t),\ x(0) = 1.$ . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . (1p)

$(c)$ Let $A \in \mathbf{R}^{n\times n}$ be any constant matrix and let $\alpha(t)$ be a continuous scalar function. What is the state transition matrix for $\dot{x}(t) = \alpha(t)Ax(t)$? . . . . . . (1p)

**2.** Consider

$$\begin{aligned} \dot{x} &= A(t)x + B(t)u \\ y &= C(t)x, \end{aligned}$$

where $x \in R^n,\ u \in R^m,\ y \in R^p$, and the dual system

$$\begin{aligned} \dot{\bar{x}} &= -A^T(t)\bar{x} + C^T(t)v \\ \bar{y} &= B^T(t)\bar{x}. \end{aligned}$$

We use $G(t,s)$ and $\bar{G}(t,s)$ to denote their impulse responses respectively. Show that $\bar{G}(t,s) = G^T(s,t)$. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . (2p)

**3.** Consider the motion of a point $x$ on a rigid body in $R^3$ with respect to the origin:

$$\dot{x} = \omega \times x + bu(t),$$

where $\omega = (1,1,1)^T$ is the angular velocity and $b = (b_1, 1, 1)^T$ is the direction of the translational velocity, $u(t)$ is free to design, and "$\times$" is the vector cross product.

$(a)$ Show that by designing $u(t)$, $x$ can reach anywhere in $R^3$ for almost all values of $b_1$. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . (3p)

$(b)$ Let $b_1 = 1,\ u = 1,\ x(0) = 0$, compute $x(t)$. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . (2p)

**4.** Consider the inverted pendulum as we did in the lecture (see Figure 1). The following equation describes the motion of the pendulum around the equilibrium $\theta = 0$:

$$L\ddot{\theta} - g\sin(\theta) + \ddot{x}\cos(\theta) = 0$$

(*a*) We consider $\ddot{x}$ as the input $u$ and $\theta$ as the output $y$. Let $x_1 = \theta$ and $x_2 = \dot{\theta}$. Derive the state space model for the linearized system (i.e. let $\sin(\theta) \approx \theta$ and $\cos(\theta) \approx 1$), and show that the model you derive is both controllable and observable. ............................................................(2p)

(*b*) Setting $u(t) = 0$ and using the linearized model, can we find an initial state $x_1(0) \neq 0$ and $x_2(0)$ such that $x_1(t) = 0$ for all $t \geq T$ where $T > 0$ is some finite time? ............................................................(2p)

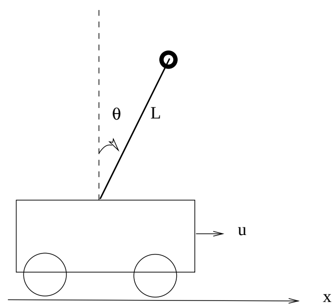
**5.** Now consider an inverted pendulum with oscillatory base (see Figure 2). The following equation describes the motion of the pendulum around the equilibrium $\theta = 0$:

$$L\ddot{\theta} - g\sin(\theta) - \ddot{z}\sin(\theta) = 0$$

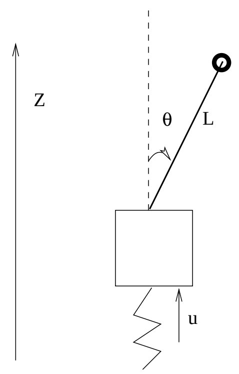
(*a*) We consider $\ddot{z}$ as the input $u$ and $\theta$ as the output $y$. Let $x_1 = \theta$ and $x_2 = \dot{\theta}$. Derive the state space model for the *linearized* system (i.e. let $\sin(\theta) \approx \theta$).(1p)

(*b*) Is the model you derive in (a) controllable? ............................ (1p)

(*c*) Let $z = 0.1\sin(\omega t)$ (be careful with the variable!). Use Matlab simulation to find out what will happen to the motion (use the original nonlinear model!) of the pendulum near $\theta = 0$ when the oscillation is “slow” and when the oscillation is “fast”. Take $L = 1$, $\theta(0) = 0.02$ and $\dot{\theta}(0) = 0$. ........................(3p)




Figur 1: Inverted pendulum on a cart.




Figur 2: Pendulum with oscillatory base.