KTH Matematik

Solution to Homework 1
Mathematical Systems Theory, SF2832
Fall 2017
You may use min(5,(your score)/4) as bonus credit on the exam.

1. Solve the following linear state equations

(a) $\dot{x}(t) = \begin{bmatrix} -1 & 1 & e^t \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} x(t),\ x(0) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ .... (2p)

Answer: $x(t) = [e^{-t} + 2te^{-t}, e^{-t}, e^{-2t}]^T$.

(b) $\dot{x}(t) = \sin(t)x(t) - \sin(t),\ x(0) = 1.$ .... (1p)

Answer: $x(t) = 1$.

(c) Let $A \in \mathbf{R}^{n\times n}$ be any constant matrix and let $\alpha(t)$ be a continuous scalar function. What is the state transition matrix for $\dot{x}(t) = \alpha(t)Ax(t)$? ...... (1p)

Answer: $\Phi(t,s) = e^{A\int_s^t \alpha(r)dr}$.

2. Consider

$$\begin{aligned} \dot{x} &= A(t)x + B(t)u \\ y &= C(t)x, \end{aligned}$$

where $x \in R^n,\ u \in R^m,\ y \in R^p$, and the dual system

$$\begin{aligned} \dot{\bar{x}} &= -A^T(t)\bar{x} + C^T(t)v \\ \bar{y} &= B^T(t)\bar{x}. \end{aligned}$$

We use $G(t,s)$ and $\bar{G}(t,s)$ to denote their impulse responses respectively. Show that $\bar{G}(t,s) = G^T(s,t)$. .... (2p)

Answer: Since $\frac{\partial}{\partial s}\Phi(t,s) = -\Phi(t,s)A(s)$, $\frac{\partial}{\partial s}\Phi^T(t,s) = -A^T(s)\Phi^T(t,s)$. Thus, $\bar{\Phi}(s,t) = \Phi^T(t,s)$. The rest follows.

3. Consider the motion of a point x on a rigid body in R^3 with respect to the origin:

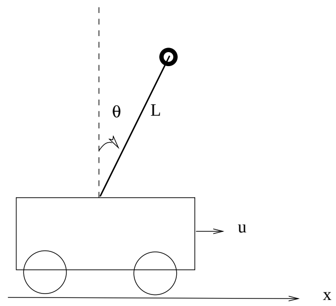
$$\dot{x} = \omega \times x + bu(t),$$

where $\omega = (1,1,1)^T$ is the angular velocity and $b = (b_1,1,1)^T$ is the direction of the translational velocity, $u(t)$ is free to design, and "×" is the vector cross product.

(*a*) Show that by designing $u(t)$, x can reach anywhere in R^3 for almost all values of b_1. . . . . (3p)
Answer: From $\omega \times x = Ax$, it is easy to see that $Ab = \omega \times b$, and $A^2 b = \omega \times (\omega \times b)$. Thus $b, Ab, A^2 b$ are linearly independent as long as b is neither orthogonal nor parallel to ω.

(*b*) Let $b_1 = 1,\ u = 1,\ x(0) = 0$, compute $x(t)$.
Answer: $b_1 = 1$ means that b is parallel to ω, thus $e^{At} b = b$ (use Taylor expansion on e^{At}). Then $x(t) = bt$. . . . . (2p)

4. Consider the inverted pendulum as we did in the lecture (see Figure 1). The following



Figur 1: Inverted pendulum on a cart.

equation describes the motion of the pendulum around the equilibrium $\theta = 0$:

$$L\ddot{\theta} - g\sin(\theta) + \ddot{x}\cos(\theta) = 0$$

Answer: Omitted.

(*a*) We consider $\ddot{x}$ as the input u and θ as the output y. Let $x_1 = \theta$ and $x_2 = \dot{\theta}$. Derive the state space model for the linearized system (i.e. let $\sin(\theta) \approx \theta$ and $\cos(\theta) \approx 1$), and show that the model you derive is both controllable and observable. . . . . (2p)

(*b*) Setting $u(t) = 0$ and using the linearized model, can we find an initial state $x_1(0) \neq 0$ and $x_2(0)$ such that $x_1(t) = 0$ for all $t \geq T$ where $T > 0$ is some finite time? . . . . (2p)

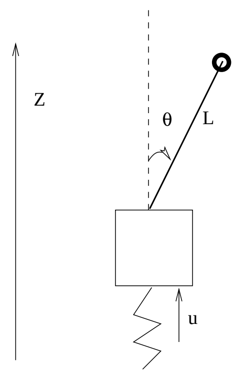
5. Now consider an inverted pendulum with oscillatory base (see Figure 2). The following equation describes the motion of the pendulum around the equilibrium $\theta = 0$:

$$L\ddot{\theta} - g\sin(\theta) - \ddot{z}\sin(\theta) = 0$$

Answer: Omitted.

(*a*) We consider $\ddot{z}$ as the input u and θ as the output y. Let $x_1 = \theta$ and $x_2 = \dot{\theta}$. Derive the state space model for the *linearized* system (i.e. let $\sin(\theta) \approx \theta$).(1p) (*b*) Is the model you derive in (a) controllable? .... (1p)

(*c*) Let $z = 0.1\sin(\omega t)$ (be careful with the variable!). Use Matlab simulation to find out what will happen to the motion (use the original nonlinear model!) of the pendulum near $\theta = 0$ when the oscillation is "slow" and when the oscillation is "fast". Take $L = 1,\ \theta(0) = 0.02$ and $\dot{\theta}(0) = 0$. .... (3p)



Figur 2: Pendulum with oscillatory base.