KTH Matematik

Solution to Homework 3
Mathematical Systems Theory, SF2832
Fall 2018
You may use min(5,(your score)/5) as bonus credit on the exam.

1. Consider

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 2 & 1 \end{bmatrix} x.$$

Answer: omitted.

(a) Is the system observable? . . . . (1p)

(b) Design a feedback controller $u = kx$ such that the closed-loop poles are placed in $\{-1, -2\}$. . . . . (2p)

(c) Is the resulting closed-loop system observable? Why? . . . . (2p)

(d) Assume now that only the output is available. Can we design an observer based controller that stabilizes the system, with the closed-loop poles located at $\{-1, -2\}$ and the observer dynamics having poles at $\{-1, -1\}$? . . . . (2p)

2. Consider

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = \gamma u$$
$$x(0) = x_0,$$

where $\gamma > 0$ can be viewed as a control gain.

(a) Design a stabilizing feedback control $u = k_1 x + k_2 x_2$ that is also the optimal control to

$$\min_u \int_0^\infty (x_1^2(t) + u^2(t))dt.$$

. . . . (3p)

Answer: By solving the ARE, we have $u^* = x_1 - \sqrt{\frac{2}{\gamma}} x_2$.

(b) What is the optimal cost as $\gamma \to \infty$? . . . . (2p)

Answer: 0.

3. Consider

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = ax_1 + u,$$

and the cost function

$$J = \int_0^{t_1} (x_2^2 + u^2)dt,$$

where a is a constant.

(a) Find the optimal control for the case $a = 0$. .... (2p)
Answer: When $a = 0$, the dynamics of x_2 is decoupled from that of x_1. Thus, for the solution P to the ARE, we have $p_1 = p_2 = 0$, while p_3 can be solved by considering only $\dot{x}_2 = u$.

(b) Let $t_1 = \infty$, discuss for what a we have a positive definite solution to the corresponding algebraic Riccati equation. .... (2p)
Answer: $a \neq 0$.

4. Let $P(t, t_f)$ denote the solution at time t with terminal time t_f to the following matrix Riccati equation:

$$\dot{P} = -A^T P - PA + PBB^T P - Q$$
$$P(t_f, t_f) = 0,$$

where A, B, Q are constant matrices and $Q \geq 0$. Show that $P(t_1, t_f) - P(t_2, t_f) \geq 0$ for any $t_1 \leq t_2 \leq t_f$. .... (3p)

Answer: When $t_0 < T_1 \leq T_2$, we have $x_0^T P(t_0, T_1)x_0 = \min_u \int_{t_0}^{T_1} (x^T Qx + u^T u)dt \leq \min_u \int_{t_0}^{T_2} (x^T Qx + u^T u)dt = x_0^T P(t_0, T_2)x_0$, namely $P(t_0, T_1) \leq P(t_0, T_2)$. However, $P(t_0, T) = P(0, T - t_0)$, the rest follows if we identify t_f as T and t_1, t_2 as t_0 in the last equality.

5. Consider the discrete time system

$$x(t + 1) = Ax(t) + Bv(t),$$
$$y(t) = Cx(t) + Dw(t),$$

where

$$x(0) = x_0, \qquad \text{E}\,[v(t)v^T(s)] = \delta_{t,s}I,$$
$$\text{E}\,[x(0)x^T(0)] = P_0, \qquad \text{E}\,[w(t)w^T(s)] = \delta_{t,s}I,$$

and where $v, w, x(0)$ are uncorrelated with zero mean value.

(a) Determine a recursive equation for the covariance matrix $V(t) = E\{x(t)x(t)^T\}$. .... (2p)

Answer: The covariance matrix $V(t) = Ex(t)x(t)^T$ has the initial condition $V(0) = P_0$, and it evolves over time as

$$\begin{aligned} V(t+1) &= Ex(t+1)x(t+1)^T = E\{(Ax(t) + Bv(t))(x(t)^T A^T + v(t)^T B^T)\} \\ &= AV(t)A^T + BB^T. \end{aligned}$$

(b) Let $\hat{x}(t) = Ex(t)$ be the Kalman estimate defined by

$$\hat{x}(t+1) = A\hat{x}(t) + AK(t)(y(t) - C\hat{x}(t))$$

and let $K(t) = P(t)C^T[CP(t)C^T + DD^T]^{-1}$ be the Kalman gain. Determine a recursive equation for the covariance matrix $P(t) = E\{(x(t) - \hat{x}(t))(x(t) - \hat{x}(t))^T\}$ of the estimation error, where we choose $P(0) = P_0$. .... (1p)
Answer: $P(t+1) = AP(t)A^T + BB^T - AP(t)C^T[CP(t)C^T + DD^T]^{-1}CP(t)A^T$.

(c) Show that $R(t) := V(t) - P(t)$ is positive semi-definite for all t .... (3p)
Answer: We prove it inductively. We have $R(0) = V(0) - P(0) = P_0 - P_0 = 0$, which is positive semidefinite. Now assume $R(t) \geq 0$ and try to show that $R(t+1) \geq 0$.

$$\begin{aligned} R(t+1) =& V(t+1) - P(t+1) = A\underbrace{(V(t) - P(t))}_{R(t)\geq 0}A^T \\ &+ \underbrace{AP(t)C^T[CP(t)C^T + DD^T]^{-1}CP(t)A^T}_{\geq 0} \end{aligned}$$

so $R(t+1)$ is positive semidefinite. $R(t)$ is now positive semi-definite for all t by induction.