KTH Matematik

Homework 1
Mathematical Systems Theory, SF2832
Fall 2018
You may use min(5,(your score)/4) as bonus credit on the exam.

1. (*a*) Solve $\dot{x}(t) = \begin{bmatrix} -1 & 1 & t \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x(t)$, $x(0) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ .... (2p)

(*b*) Solve $\dot{x}(t) = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} x(t)$, $x(0) = \begin{bmatrix} x_1^0 \\ x_2^0 \end{bmatrix}$, where ω is a positive constant, and show that $x_1(t) = a\sin(\omega t + b)$ where $a,\ b$ are functions of $x(0)$. ....(2p)

2. Consider an $n \times n$ matrix

$$A(t) \quad = \quad e^{A_1 t} A_2 e^{-A_1 t},$$

where $A_1,\ A_2$ are constant matrices.

(*a*) Show that $\Psi(t) = e^{A_1 t} e^{A_2 t} e^{-A_1 t}$ is a fundamental matrix to $\dot{x} = A(t)x$ if $A_1 A_2 = A_2 A_1$. .... (2p)

(*b*) For the general case, find a fundamental matrix in the form $\Psi(t) = e^{A_1 t} e^{A_3 t}$ and specify your A_3 in terms of $A_1,\ A_2$. ....(2p)

3. Consider

$$\begin{aligned} \dot{x} &= A(t)x + B(t)u \\ y &= C(t)x. \end{aligned}$$

Show that controllability and observability of linear time-varying systems are invariant under linear transformation $\bar{x} = P(t)x$, where $P(t)$ is nonsingular and continuously differentiable for all $t \in (-\infty, \infty)$. ....(4p)

4. Consider the inverted pendulum as we did in the lecture (see Figure 1). The following equation describes the motion of the pendulum around the equilibrium $\theta = 0,\ \dot{\theta} = 0$:

$$L\ddot{\theta} - g\sin(\theta) + \ddot{x}\cos(\theta) = 0$$

(*a*) We consider $\ddot{x}$ as the input u and θ as the output y. Let $x_1 = \theta$ and $x_2 = \dot{\theta}$. Derive the state space model for the linearized system (i.e. let $\sin(\theta) \approx \theta$ and $\cos(\theta) \approx 1$), and show that the model you derive is both controllable and observable. ....(2p)

(b) Setting $u(t) = 0$ and using the linearized model, can we find an initial state $x_1(0) \neq 0$ and $x_2(0)$ such that $x_1(t) = 0$ for all $t \geq T$ where $T > 0$ is some finite time? . . . . . . (2p)

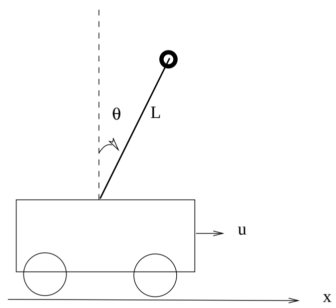
5. Now consider an inverted pendulum with oscillatory base (see Figure 2). The following equation describes the motion of the pendulum around the equilibrium $\theta = 0,\ \dot{\theta} = 0$:

$$L\ddot{\theta} - g\sin(\theta) - \ddot{z}\sin(\theta) = 0$$

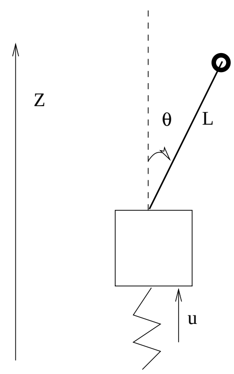
(a) We consider $\ddot{z}$ as the input u and θ as the output y. Let $x_1 = \theta$ and $x_2 = \dot{\theta}$. Derive the state space model for the *linearized* system. . . . . . . (1p)

(b) Is the model you derive in (a) controllable? . . . . . . (1p)

(c) Let $z = 0.1\sin(\omega t)$ (be careful with the variable!). Use Matlab simulation to find out what will happen to the motion (use the original nonlinear model!) of the pendulum near $\theta = 0$ when the oscillation is "slow" and when the oscillation is "fast". Take $L = 1,\ \theta(0) = 0.02$ and $\dot{\theta}(0) = 0$. . . . . . . (3p)



Figur 1: Inverted pendulum on a cart.



Figur 2: Pendulum with oscillatory base.