Realization Theory, Part III

1. Theory overview (recap from last time).
2. Equivalent realizations.
3. Ho's algorithm.
   - Alternative proof (optional)
4. Realization theory for discrete time systems
5. Kalman's experiment revisited (optional)

Theory overview

Given a strictly proper rational transfer matrix $R(s)$ such that

$$R(s) = \frac{1}{\chi(s)}\left(N_0 + N_1 s + \dots N_{r-1}s^{r-1}\right)$$
$$= R_1 s^{-1} + R_2 s^{-2} + R_3 s^{-3}\dots$$
$$\chi(s) = s^r + a_1 s^{r-1} + \dots + a_r$$ (is the least common denominator of the elements $R_{ij}(s)$ of $R(s)$)

the block Hankel matrix is defined from the Markov parameters as

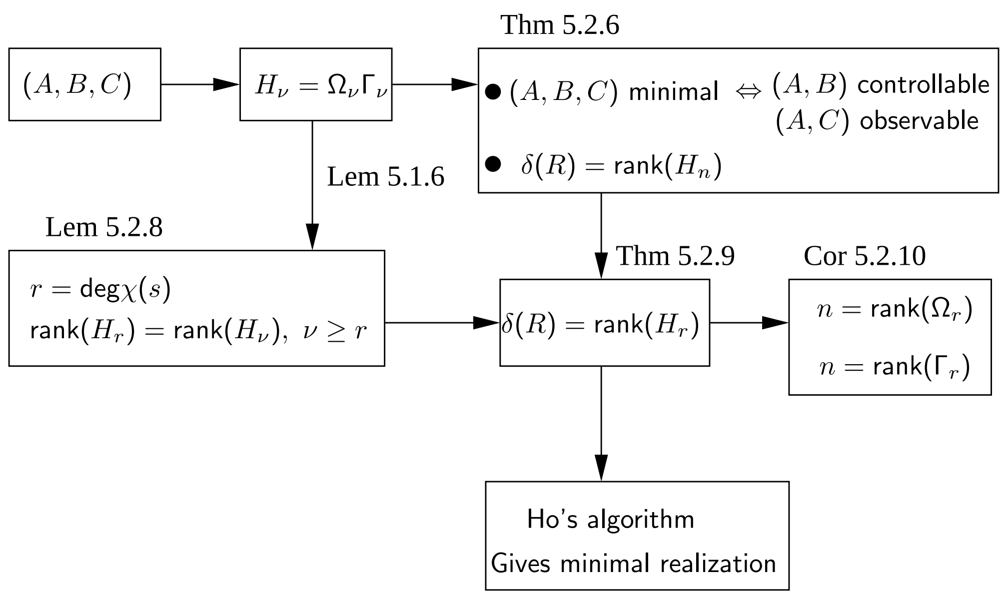
$$H_\nu = \begin{bmatrix} R_1 & R_2 & \dots & R_\nu \\ R_2 & R_3 & \dots & R_{\nu+1} \\ \vdots & & & \vdots \\ R_\nu & R_{\nu+1} & \dots & R_{2\nu-1} \end{bmatrix}$$

Given any realization (A, B, C), the Markov parameters are given as $R_k = CA^{k-1}B$. This implies that

$$H_r = \begin{bmatrix} CB & CAB & \dots & CA^{\nu-1}B \\ CAB & CA^2B & \dots & CA^\nu B \\ \vdots & & \dots & \vdots \\ CA^{\nu-1}B & CA^\nu B & \dots & CA^{2\nu-2}B \end{bmatrix}$$
$$= \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{\nu-1} \end{bmatrix} \begin{bmatrix} B & AB & \dots & A^{\nu-1}B \end{bmatrix} = \Omega_\nu \Gamma_\nu$$

If $(\tilde{A}, \tilde{B}, \tilde{C})$ is another realization then $H_\nu = \tilde{\Omega}_\nu \tilde{\Gamma}_\nu = \Omega_\nu \Gamma_\nu$

Overview of Theory



Main points for the user

- The MacMillan degree $\delta(R)$, i.e. the dimension of any minimal realization, can be determined from the Hankel matrix as $\delta(R) = \text{rank}\,(H_r)$.
- Any minimal realization is completely reachable and completely observable.
- There always exists a minimal realization. Indeed, we already know how to construct a standard reachable realization. If this realization is not observable then we can remove the unobservable states using a Kalman decomposition. This gives a minimal realization.

Equivalent Realizations

Consider a change of basis

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases} \quad \overset{z=Tx}{\Leftrightarrow} \quad \begin{cases} \dot{z} = \tilde{A}z + \tilde{B}u \\ u = \tilde{C}z \end{cases}$$

where $\tilde{A} = TAT^{-1}$, $\tilde{B} = TB$, $\tilde{C} = CT^{-1}$.

Invariants under change of basis (state space transformation)

- The systems eigenvalues are invariant under a change of basis since

$$\chi(\tilde{A})(s) = \det(sI - \tilde{A}) = \det(sI - \tilde{A}) = \chi(A)(s)$$

- The system transfer function is invariant under change of basis

$$\tilde{G}(s) = \tilde{C}(sI - \tilde{A})^{-1}\tilde{B} = C(sI - A)^{-1}B = G(s)$$

The Isomorphism Theorem

Let (A, B, C) and $(\tilde{A}, \tilde{B}, \tilde{C})$ be two minimal realizations of a strictly proper transfer function. Then there exists an invertible matrix T such that

$$\tilde{A} = TAT^{-1}, \tilde{B} = TB, \tilde{C} = CT^{-1}$$

If

$$\Gamma = \begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix}, \quad \Omega = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}, \quad \tilde{\Omega} = \begin{bmatrix} \tilde{C} \\ \tilde{C}\tilde{A} \\ \vdots \\ \tilde{C}\tilde{A}^{n-1} \end{bmatrix}$$

$$\tilde{\Gamma} = \begin{bmatrix} \tilde{B} & \tilde{A}\tilde{B} & \dots & \tilde{A}^{n-1}\tilde{B} \end{bmatrix}$$

then $T = \tilde{\Gamma}\Gamma^T(\Gamma\Gamma^T)^{-1} = (\tilde{\Omega}^T\tilde{\Omega})^{-1}\tilde{\Omega}^T\Omega$.

How to obtain minimal realizations

Three ways to obtain a minimal realization from a strictly proper $R(s)$

$$R(s) = \frac{1}{\chi(s)}\left(N_0 + N_1 s + \ldots + N_{r-1}s^{r-1}\right)$$
$$= R_1 s^{-1} + R_2 s^{-2} + R_3 s^{-3} + \ldots \quad (R_k \text{ is the } k^{th} \text{ Markov parameter})$$

where

$$\chi(s) = s^r + a_1 s^{r-1} + \ldots + a_r, \quad \text{(Least common denominator)}$$

1. Standard reachable realization + Kalman decomposition
2. Standard observable realization + Kalman decomposition
3. Ho's algorithm

The shift of a Hankel matrix

$$H_r = \begin{bmatrix} R_1 & R_2 & \dots & R_r \\ R_2 & R_3 & \dots & R_{r+1} \\ \vdots & & & \vdots \\ R_r & R_{r+1} & \dots & R_{2r-1} \end{bmatrix}$$

is defined as

$$\sigma(H_r) = \begin{bmatrix} R_2 & R_3 & \dots & R_{r+1} \\ R_3 & R_4 & \dots & R_{r+2} \\ \vdots & & & \vdots \\ R_{r+1} & R_{r+2} & \dots & R_{2r+1} \end{bmatrix}$$

Ho's algorithm

1. Let $r = \deg(\chi(s))$
2. Determine invertible matrices such that $PH_rQ = \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix}$
3. Let $n = \delta(R) = \text{rank}\,(H_r)$ (the McMillan degree)
4. A minimal realization is obtained as

$$A = \begin{bmatrix} I_n & 0 \end{bmatrix} P\sigma(H_r)Q \begin{bmatrix} I_n \\ 0 \end{bmatrix} \qquad B = \begin{bmatrix} I_n & 0 \end{bmatrix} P \begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_r \end{bmatrix} \tag{1}$$

$$C = \begin{bmatrix} R_1 & R_2 & \dots & R_r \end{bmatrix} Q \begin{bmatrix} I_n \\ 0 \end{bmatrix}$$

Alternative Proof of Ho's Algorithm (optional)

It is not so difficult to understand why Ho's algorithm works but the full proof is complicated. We will here try to briefly explain an alternative proof, which provides some further intuition.

Remark:The main part of the proof in Lindquist and Sand is to show that the realization formulas in Ho's algorithm, formula (1), in fact gives a realization (A, B, C), i.e. they show that $CA^{k-1}B = R_k$. This is fairly complicated and involves manipulation with so-called pseudo inverses.

We know that there exists a $n = \delta(R) = \text{rank}\,(H_r)$-dimensional (minimal) realization. This implies that there exists a realization $(\tilde{A}, \tilde{B}, \tilde{C})$ with $\tilde{A} \in \mathbf{R}^{n\times n}$. The Hankel matrix can now be factorized as $H_r = \tilde{\Omega}_r\tilde{\Gamma}_r$, where

$$\tilde{\Omega}_r = \begin{bmatrix} \tilde{C} \\ \tilde{C}\tilde{A} \\ \vdots \\ \tilde{C}\tilde{A}^{r-1} \end{bmatrix} \qquad \tilde{\Gamma}_r = \begin{bmatrix} \tilde{B} & \tilde{A}\tilde{B} & \dots & \tilde{A}^{r-1}\tilde{B} \end{bmatrix}$$

This is a full rank factorization because $\text{Im}\,\tilde{\Gamma}_r = \mathbf{R}^n$ and $\text{Ker}\,\tilde{\Omega}_r = 0$, and thus $\text{rank}\,(\tilde{\Omega}_r) = \text{rank}\,(\tilde{\Gamma}_r) = n$ (Theorem 5.2.9).

If we use that $\sigma(H_r) = \tilde{\Omega}_r \tilde{A} \tilde{\Gamma}_r$ then it is easy to establish that

$$\tilde{A} = \tilde{\Omega}_r^\dagger \sigma(H_r) \tilde{\Gamma}_r^\dagger$$
$$\tilde{B} = \tilde{\Omega}_r^\dagger H_r \begin{bmatrix} I_m \\ 0 \end{bmatrix} \qquad (2)$$
$$\tilde{C} = \begin{bmatrix} I_q & 0 \end{bmatrix} H_r \tilde{\Gamma}_r^\dagger$$

where $\tilde{\Omega}_r^\dagger = (\tilde{\Omega}_r^T \tilde{\Omega}_r)^{-1} \tilde{\Omega}_r^T$ is a left inverse of $\tilde{\Omega}_r$, i.e. $\tilde{\Omega}_r^\dagger \tilde{\Omega}_r = I_n$ and $\tilde{\Gamma}_r^\dagger = \tilde{\Gamma}_r^T (\tilde{\Gamma}_r \tilde{\Gamma}_r^T)^{-1}$ is a right inverse of $\tilde{\Gamma}_r$, i.e. $\tilde{\Gamma}_r \tilde{\Gamma}_r^\dagger = I_n$. For example, the second formula follows since

$$\tilde{\Omega}_r^\dagger H_r \begin{bmatrix} I_m \\ 0 \end{bmatrix} = (\tilde{\Omega}_r^T \tilde{\Omega}_r)^{-1} \tilde{\Omega}_r^T \tilde{\Omega}_r \tilde{\Gamma}_r \begin{bmatrix} I_m \\ 0 \end{bmatrix} = \tilde{B}$$

From the second step Ho's algorithm we get the factorization

$$H_r = \underbrace{P^{-1} \begin{bmatrix} I_n \\ 0 \end{bmatrix}}_{\Omega_r} \underbrace{\begin{bmatrix} I_n & 0 \end{bmatrix} Q^{-1}}_{\Gamma_r} =: \Omega_r \Gamma_r \qquad (3)$$

This is a full rank factorization, i.e. $\text{rank}\,(\Gamma_r) = \text{rank}\,(\Omega_r) = n$. If we could establish that Ω_r is an observability matrix and Γ_r is a reachability matrix corresponding to a minimal realization (A, B, C), i.e. if

$$\Omega_r = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{r-1} \end{bmatrix}, \qquad \Gamma_r = \begin{bmatrix} B & AB & \ldots & A^{r-1}B \end{bmatrix} \qquad (4)$$

then we could proceed as above and use (2) to extract this realization.

In this case we can use (3) to derive the following left and right inverses

$$\Omega_r^\dagger = \begin{bmatrix} I_n & 0 \end{bmatrix} P$$
$$\Gamma_r^\dagger = Q \begin{bmatrix} I_n \\ 0 \end{bmatrix}$$

The formulas in (2) with the above left and right inverses corresponds to (1). This means that the proof of Ho's algorithm is complete if we can show that Ω_r and Γ_r indeed has the form (4). To do this we note that any two full rank factorizations are related via

$$H_r = \Omega_r \Gamma_r = \Omega_r T^{-1} T \Gamma_r = \tilde{\Omega}_r \tilde{\Gamma}_r$$

for some invertible $n \times n$ matrix.

Suppose $\Omega_r = \begin{bmatrix} F_1 \\ F_2 \\ \vdots \\ F_r \end{bmatrix}$ for some $F_k \in \mathbf{R}^{q \times n}$. From above we have

$$\Omega_r T^{-1} = \begin{bmatrix} F_1 \\ F_2 \\ \vdots \\ F_r \end{bmatrix} T^{-1} = \begin{bmatrix} \tilde{C} \\ \tilde{C}\tilde{A} \\ \vdots \\ \tilde{C}\tilde{A}^{r-1} \end{bmatrix}$$

The first equation shows that $F_1 = \tilde{C}T$. If we introduce the notation $C = \tilde{C}T$ and use this in the second block, we get

$$F_2 = \tilde{C}\tilde{A}T = \tilde{C}TT^{-1}\tilde{A}T = CA, \quad \text{where we defined } A = T^{-1}\tilde{A}T$$

If we continue like this and similarly for Γ_r we see that Ω_r and Γ_r indeed has the form (4), where

$$A = T^{-1}\tilde{A}T, \qquad B = T^{-1}\tilde{B}, \qquad C = \tilde{C}T$$

This shows that the realization corresponding to the factorization $H_r = \tilde{\Omega}_r\tilde{\Gamma}_r$ is related to the realization corresponding to the factorization $H_r = \Omega_r\Gamma_r$ via a coordinate change. This is the essence of the state space isomorphism theorem.

Comments

- From a numerical point of view it is preferable to use the singular value decomposition in step 2 of Ho's algorithm to obtain a full rank factorization.
- If the Hankel matrix is obtained from experimental data then this technique of computing the system realization is called subspace identification. There are many variations on the same idea.

Realization theory for discrete time systems

Consider the problem of finding a discrete time realization

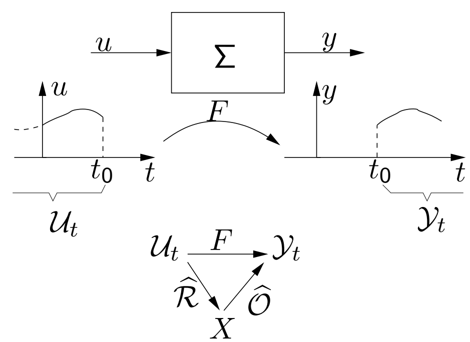
$$x_{k+1} = Ax_k + Bu_k$$
$$y_k = Cx_k$$

of a strictly proper rational transfer function $R(z)$, i.e. to find (A, B, C) of minimal dimensions such that

$$R(z) = C(zI - A)^{-1}B$$

This problem can be solved using the same results and the same algorithms as we used for the continuous time systems (we can, for example, use Ho's algorithm).

Kalman's Experiment Revisited (optional)



- Factorize as $F = \hat{\mathcal{O}}\hat{\mathcal{R}}$.
- *Reachability operator* $\hat{\mathcal{R}}$ summarizes prior inputs in the state.
- *Observability operator* $\hat{\mathcal{O}}$ generates future outputs using the state.
- The dimension of the state space X should be as small as possible in order to obtain maximal data reduction.

Discrete time systems

$$l_2(Z_-) \longrightarrow \boxed{\Sigma : \begin{cases} x_{k+1} &= Ax_k + Bu_k \\ y_k &= Cx_k \end{cases}} \longrightarrow l_2(Z_+)$$

Formal calculations shows (with $t_0 = 0$)

$$\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \end{bmatrix} = \underbrace{\begin{bmatrix} CB & CAB & CA^2B & \dots \\ CAB & CA^2B & CA^3B & \dots \\ CA^2B & CA^3B & CA^4B & \dots \\ \vdots & \vdots & \vdots & \end{bmatrix}}_{H} \begin{bmatrix} u_{-1} \\ u_{-2} \\ u_{-3} \\ \vdots \end{bmatrix}$$

The block *Hankel* matrix can be factorized as

$$H = \begin{bmatrix} CB & CAB & CA^2B & \dots \\ CAB & CA^2B & CA^3B & \dots \\ CA^2B & CA^3B & CA^4B & \dots \\ \vdots & \vdots & \vdots & \end{bmatrix} = \underbrace{\begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \end{bmatrix}}_{\Omega_\infty} \underbrace{\begin{bmatrix} B & AB & A^2B & \dots \end{bmatrix}}_{\Gamma_\infty}$$

$$\sim \begin{bmatrix} \Omega_r \\ 0 \end{bmatrix} \begin{bmatrix} \Gamma_r & 0 \end{bmatrix} = \begin{bmatrix} H_r & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \tilde{\Omega}_r \\ 0 \end{bmatrix} \begin{bmatrix} \tilde{\Gamma}_r & 0 \end{bmatrix} \sim \tilde{\Omega}_\infty \tilde{\Gamma}_\infty$$

- The equivalences are due to the Cayley-Hamilton Theorem
- $H_r = \tilde{\Omega}_r \tilde{\Gamma}_r$ is a full rank factorization, i.e $\text{Im}(\tilde{\Gamma}_r) = \mathbf{R}^n$ and $\text{Ker}\,(\tilde{\Omega}) = \{0\}$.

- $H = \tilde{\Omega}_\infty \tilde{\Gamma}_\infty$ is the desired factorization into a reachability and observability operator that provides the maximal data reduction. It can be implemented using a minimal state space realization corresponding to $\tilde{\Omega}_r$ and $\tilde{\Gamma}_r$

$$x_{k+1} = \tilde{A}x_k + \tilde{B}u_k$$
$$y_k = \tilde{C}x_k$$

Commutative diagram illustrating the two factorizations