State Feedback and Observer Based Control

1. State feedback control

   Multivariable case

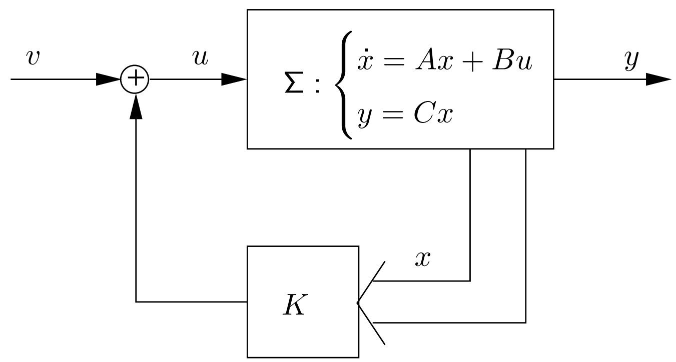
2. Observer based control and observer design
3. When is stabilization possible?
   (a) The state feedback case

   If the system is stabilizable

   (b) Output feedback case with observer based controller

   If the system is stabilizable and detectable

State Feedback



- How do we change the system properties by state feedback?
  - Pole placement of multivariable systems.

Multivariable Case

There exists several systematic approaches

1. Use multivariable generalization of the controller form
   - You do not need to know how to derive the form.
   - You need to know how to use it.
2. Transform to the single input case (see Lindquist and Sand) (optional)
3. Use Lypaunov equations (beyond the scope of the course)

Suppose (A, B) is completely controllable, where $B = \begin{bmatrix} b_1 & b_2 & \dots & b_m \end{bmatrix}$. Follow the following construction

1. Find the largest index r_1 such that $\{b_1, Ab_1, \dots, A^{r_1-1}b_1\}$ are linearly independent (this means that $A^{r_1}b_1$ is linearly dependent of these vectors).
2. Find largest r_2 such that $\{b_1, Ab_1, \dots, A^{r_1-1}b_1, b_2, Ab_2, \dots A^{r_2-1}b_2\}$ are linearly independent
3. Continue this process until we have found indices $r_1, \dots, r_m$ such that

   $$\hat{\Gamma} = \begin{bmatrix} b_1 & Ab_1 & \dots & A^{r_1-1}b_1 & b_2 & Ab_2 & \dots A^{r_2-1}b_2 & \dots & A^{r_m-1}b_m \end{bmatrix}$$

   is an invertible $n \times n$ matrix.

4. Find row vectors t_k, $k = 1, \dots, m$ such that

$$\begin{bmatrix} t_1 \\ t_2 \\ \vdots \\ t_m \end{bmatrix} \hat{\Gamma} = \left[\begin{array}{cccc|cccc|c|cccc} 0 & 0 & \dots & 1 & 0 & 0 & \dots & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 1 & \dots & 0 & 0 & \dots & 0 \\ & & & & & & \vdots & & & & & & \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & \dots & 0 & 0 & \dots & 1 \end{array}\right] \qquad (1)$$

where the 1 are at position r_1, $r_1 + r_2$ and so on. There is a unique solution to this equation because $\hat{\Gamma}$ is invertible.

5. Define the matrix

$$T = \begin{bmatrix} t_1 \\ t_1 A \\ \vdots \\ t_1 A^{r_1 - 1} \\ t_2 \\ t_2 A \\ \vdots \\ t_2 A^{r_2 - 1} \\ \vdots \\ t_m A^{r_m - 1} \end{bmatrix}$$

It can be proven that T is invertible. The proof is analogous to the SISO case in the lecture notes and it makes use of (1).

6. The change of coordinates $F = TAT^{-1}$ and $H = TB$ gives

$$F = \left[\begin{array}{cccc|cccc|c|cccc} 0 & 1 & \dots & 0 & 0 & 0 & \dots & 0 & \dots & 0 & 0 & 0 & 0 \\ & & \ddots & & & & & & & & & & \\ 0 & 0 & \dots & 1 & 0 & 0 & \dots & 0 & \dots & 0 & 0 & 0 & 0 \\ * & * & \dots & * & * & * & \dots & * & \dots & * & * & * & * \\ \hline 0 & 0 & \dots & 0 & 0 & 1 & \dots & 0 & \dots & 0 & 0 & 0 & 0 \\ & & & & & & \ddots & & & & & & \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 1 & \dots & 0 & 0 & 0 & 0 \\ * & * & \dots & * & * & * & \dots & * & \dots & * & * & * & * \\ \hline & & \vdots & & & & \vdots & & \dots & & & \vdots & \\ \hline 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & \dots & 0 & 1 & 0 & 0 \\ & & & & & & & & & & & \ddots & \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & \dots & 0 & 0 & 0 & 1 \\ * & * & \dots & * & * & * & \dots & * & \dots & * & * & * & * \end{array}\right] \quad H = \left[\begin{array}{cccc} 0 & 0 & \dots & 0 \\ 0 & 0 & & 0 \\ & & \vdots & \\ 1 & 0 & \dots & 0 \\ \hline 0 & 0 & \dots & 0 \\ 0 & 0 & & 0 \\ & & \vdots & \\ 0 & 1 & \dots & 0 \\ \hline & & \vdots & \\ \hline 0 & 0 & \dots & 0 \\ 0 & 0 & & 0 \\ & & \vdots & \\ 0 & 0 & \dots & 1 \end{array}\right]$$

7. In the new coordinates consider a state feedback $u = Gz$ such that

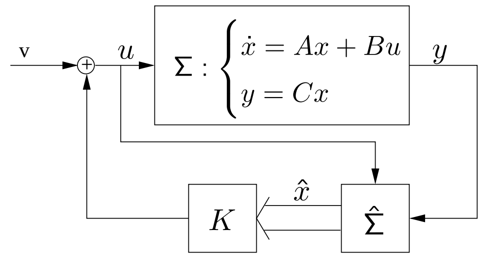
$$F + HG = \left[\begin{array}{cccc|cccc|c|cccc} 0 & 1 & \dots & 0 & 0 & 0 & \dots & 0 & \dots & 0 & 0 & \dots & 0 \\ & & \ddots & & & & & & & & & & \\ 0 & 0 & \dots & 1 & 0 & 0 & \dots & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 1 & 0 & \dots & 0 & \dots & 0 & 0 & \dots & 0 \\ \hline 0 & 0 & \dots & 0 & 0 & 1 & \dots & 0 & \dots & 0 & 0 & \dots & 0 \\ & & & & & & \ddots & & & & & & \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 1 & \dots & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & \dots & 0 & 0 & \dots & 0 \\ \hline & & \vdots & & & & \vdots & & \dots & & & \vdots & \\ \hline 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & \dots & 0 & 1 & \dots & 0 \\ & & & & & & & & & & & \ddots & \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & \dots & 0 & 0 & \dots & 1 \\ -\gamma_n & -\gamma_{n-1} & \dots & -\gamma_{n-\mu_1} & * & * & \dots & * & & * & * & \dots & -\gamma_1 \end{array}\right]$$

8. If we let $K = GT$ then

$$\begin{aligned}\chi_{A+BK}(s) &= \det(sI - A - BK)\\ &= \det(sI - T^{-1}FT - T^{-1}HGT)\\ &= \det(T^{-1})\det(s - F - HG)\det(T)\\ &= \det(s - F - HG) = s^n + \gamma_1 s^{n-1} + \ldots + \gamma_n\end{aligned}$$

which is the desired characteristic polynomial.

Output feedback/Observer based control



- The role of the observer is to estimate the state.
- How to construct an observer, $\hat{\Sigma}$, with desired convergence properties for the estimation error $e(t) = x(t) - \hat{x}(t) \rightarrow 0$.
- Use the state estimate for state feedback $u = K\hat{x} + v$.

Observer

The observer consists of a copy of the plant and a term that compensates for the estimation error

$$\hat{\Sigma} : \dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x}) \qquad (2)$$

The estimation error $e = x - \hat{x}$ satisfies

$$\dot{e} = (A - LC)e$$

- The observer poles, i.e. the roots of $\chi_{A-LC}(s) = \det(sI - (A - LC)) = 0$, determines if and how fast the estimation error converges to zero.
- The observer poles can be assigned arbitrarily if (A, C) is observable.
- The observer design problem is dual to the state feedback design problem.

Observer

The closed loop dynamics of the observer based control system is

$$\begin{bmatrix}\dot{x}\\ \dot{e}\end{bmatrix} = \begin{bmatrix}A + BK & -BK\\ 0 & A - LC\end{bmatrix}\begin{bmatrix}x\\ e\end{bmatrix} v + \begin{bmatrix}B\\ 0\end{bmatrix} v$$

$$y = \begin{bmatrix}C & 0\end{bmatrix}\begin{bmatrix}x\\ e\end{bmatrix}$$

It follows that the closed loop eigenvalues can be assigned arbitrarily if

1. (A, B) is controllable
2. (A, C) is observable

Stabilizability

Consider the system

$$\dot{x} = Ax + Bu \tag{3}$$

Consider a Kalman decomposition

$$\begin{bmatrix} \dot{x}_r \\ \dot{x}_{\bar{r}} \end{bmatrix} = \begin{bmatrix} A_{11;r} & A_{12;r} \\ 0 & A_{22;r} \end{bmatrix} \begin{bmatrix} x_r \\ x_{\bar{r}} \end{bmatrix} + \begin{bmatrix} B_{1;r} \\ 0 \end{bmatrix} u$$

Definition 1. *The pair (A, B) is called stabilizable if $(A_{11;r}, B_{1;r})$ is controllable (completely reachable) and if $A_{22;r}$ is stable.*

Theorem 1. *The system (3) can be stabilized using a state feedback $u = Kx$ if (and only if) (A, B) is stabilizable.*

Proof. The closed loop system $\dot{x} = (A + BK)x$ can in coordinates from the Kalman decomposition be represented as $(u = K_1 x_r + K_2 x_{\bar{r}})$

$$\begin{bmatrix} \dot{x}_r \\ \dot{x}_{\bar{r}} \end{bmatrix} = \begin{bmatrix} A_{11;r} + B_{1;r}K_1 & A_{12;r} + B_{1;r}K_2 \\ 0 & A_{22;r} \end{bmatrix} \begin{bmatrix} x_r \\ x_{\bar{r}} \end{bmatrix}$$

(i) The eigenvalues of this system are the roots of

$$\begin{aligned} \chi_{A+BK}(s) &= \det(sI - A - BK) \\ &= \det(sI - A_{11;r} - B_{1;r}K_1)\det(sI - A_{22;r}) = 0 \end{aligned}$$

(ii) Since $(A_{11;r}, B_{1;r})$ is controllable we know that the eigenvalues of $A_{11;r} + B_{1;r}K_1$ can be made stable by suitable design of K_1

(iii) $A_{22;r}$ is stable

Hence, $(i) - (iii)$ implies that $A + BK$ can be made stable by suitable choice of K. □

Detectability

Consider the system

$$\begin{aligned} \dot{x} &= Ax \\ y &= Cx \end{aligned} \tag{4}$$

Consider a Kalman decomposition

$$\begin{bmatrix} \dot{x}_{\bar{o}} \\ \dot{x}_o \end{bmatrix} = \begin{bmatrix} A_{11;o} & A_{12;o} \\ 0 & A_{22;o} \end{bmatrix} \begin{bmatrix} x_{\bar{o}} \\ x_o \end{bmatrix}$$

$$y = \begin{bmatrix} 0 & C_{2;o} \end{bmatrix} \begin{bmatrix} x_{\bar{o}} \\ x_o \end{bmatrix}$$

Definition 2. *The pair (A, C) is called detectable if $(A_{22;o}, C_{2;o})$ is observable (completely observable) and if $A_{11;o}$ is stable.*

Theorem 2. *The estimation error $\dot{e} = (A - LC)e$ of an observer can be stabilized using observer gain L if (and only if) (A, C) is detectable.*

Proof. In the coordinates of the Kalman decomposition we have

$$\begin{bmatrix} \dot{e}_{\bar{o}} \\ \dot{e}_o \end{bmatrix} = \begin{bmatrix} A_{11;o} & A_{12;o} - L_1C_{2;o} \\ 0 & A_{22;o} - L_2C_{2;o} \end{bmatrix} \begin{bmatrix} e_{\bar{o}} \\ e_o \end{bmatrix}$$

(i) The eigenvalues are the roots of

$$\chi_{A-LC}(s) = \det(sI - A_{11;o})\det(sI - A_{22;o} + L_2C_{2;o}) = 0$$

(ii) $A_{11;o}$ is stable

(iii) $A_{22;o} - L_2C_{2;o}$ can be made stable by choosing L_2 suitably since $(A_{22;o}, C_{2;o})$ is observable.

Hence, $(i)-(iii)$ implies that $A-LC$ can be made stable by choosing L suitably. □

When is stabilization possible using an observer based controller?

The closed loop system is

$$\begin{bmatrix}\dot{x}\\ \dot{\hat{x}}\end{bmatrix}=\begin{bmatrix}A & BK\\ LC & A-LC+BK\end{bmatrix}\begin{bmatrix}x\\ \hat{x}\end{bmatrix}+\begin{bmatrix}B\\ B\end{bmatrix}v$$

In the new coordinates $(x,e)=(x,x-\hat{x})$ this becomes

$$\begin{bmatrix}\dot{x}\\ \dot{e}\end{bmatrix}=\begin{bmatrix}A+BK & -BK\\ 0 & A-LC\end{bmatrix}\begin{bmatrix}x\\ e\end{bmatrix}+\begin{bmatrix}B\\ 0\end{bmatrix}v$$

$$y=\begin{bmatrix}C & 0\end{bmatrix}\begin{bmatrix}x\\ e\end{bmatrix}$$

The prediction error, e, is not controllable!

Decomposition gives

$$\begin{bmatrix}\dot{x}_r\\ \dot{x}_{\bar{r}}\\ \dot{e}_{\bar{o}}\\ \dot{e}_o\end{bmatrix}=\begin{bmatrix}A_{11;r}+B_{1;r}K_1 & A_{12;r}+B_{1;r}K_2 & -B_{1;r}K_1 & -B_{1;r}K_2\\ 0 & A_{22;r} & 0 & 0\\ 0 & 0 & A_{11;o} & A_{12;o}-L_1C_{2;o}\\ 0 & 0 & 0 & A_{22;o}-L_2C_{2;o}\end{bmatrix}\begin{bmatrix}x_r\\ x_{\bar{r}}\\ e_{\bar{o}}\\ e_o\end{bmatrix}+\begin{bmatrix}B_{1;r}\\ 0\\ 0\\ 0\end{bmatrix}v \quad (5)$$

- $A_{11;r}, A_{12;r}, A_{22;r}, B_{1;r}$ is a decomposition with respect to reachability.
- $A_{11;o}, A_{12;o}, A_{22;o}, C_{2;o}$ is a decomposition with respect to observability.

Theorem 3. *The system*

$$\dot{x}=Ax+Bu$$
$$y=Cx$$

can be stabilized using observer based control if (and only if) (A,B) *is stabilizable and* (A,C) *is detectable.*

Proof. The error dynamics can be made stable by suitable choice of L_2 if $(A_{22,o},C_{2;o})$ is observable and $A_{11,o}$ is stable. Then the complete dynamics in (5) can be made stable by suitable choice of K_1 if $(A_{11;r},B_{1,r})$ is controllable and $A_{22,r}$ is stable. □