Reachability for Time-invariant Systems

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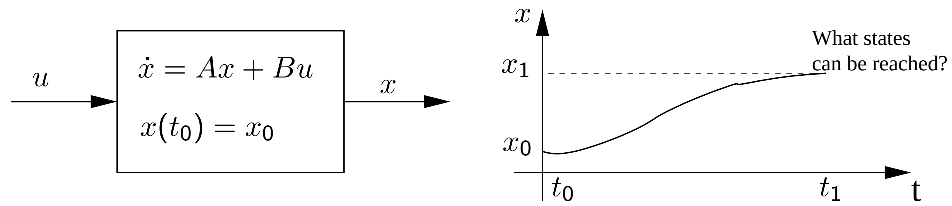
The Reachability Problem



- The state x_1 is reachable from x_0 if there exists a bounded control that in finite time transfers the state from x_0 to x_1.
- Equivalent formulation: Find $u \in \mathrm{L}_2^m[t_0, t_1]$ such that

$$Lu = d, \quad \text{where} \quad \begin{cases} Lu = \int_{t_0}^{t_1} \Phi(t_1, t)B(t)u(t)dt \\ d = x_1 - \Phi(t_1, t_0)x_0, \end{cases} \tag{1}$$

- There exists a solution to (1) if $d \in \mathrm{Im}\,(L) = \mathrm{Im}\, LL^*$
- The solution is not unique because $\mathrm{Ker}\,(L) \neq 0$.
- The minimum norm solution can be constructed as

$$u = L^*\lambda \qquad \text{where } \lambda \text{ solves } LL^*\lambda = d.$$

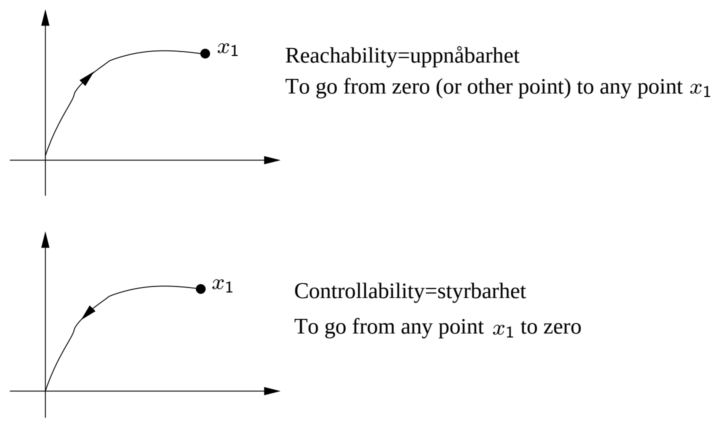
- We have

$$(L^*d)(t) = \Phi(t_1, t)^T B(t)^T d$$

$$W(t_0, t_1) = LL^* = \int_{t_0}^{t_1} \Phi(t_1, \tau)B(\tau)B^T(\tau)\Phi(t_1, \tau)^T d\tau$$

where $W(t_0, t_1)$ is the Reachability Gramian.

Terminology



- In the course we normally do not always distinguish between these definitions so reachability, controllability, (uppnåbarhet, styrbarhet) will mean that you can transfer from some point to another.

Subspace Algebra

We next summarize a few results on subspace algebra

- A subset $\mathcal{V} \subset \mathbf{R}^n$ is a *linear subspace* if $\forall v_1, v_2 \in \mathcal{V}$ and $\forall \alpha_1, \alpha_2 \subset \mathbf{R}$, $\alpha_1 v_1 + \alpha_2 v_2 \in \mathcal{V}$.

  A subspace $\mathcal{V} \subset \mathbf{R}^n$ is said to be *spanned* by the set of vectors $v_1, \ldots, v_r \in \mathcal{V}$ if every $v \in \mathcal{V}$ can be written as a linear combination of the v_k, i.e., there exists $\alpha_k \in \mathbf{R}$ such that $v = \sum_{k=1}^{r} \alpha_k v_k$. The above can in more compact notation be written

$$\mathcal{V} = \text{span}\{v_1, \ldots, v_r\} := \left\{ \sum_{k=1}^{r} \alpha_k v_k : \alpha_k \in \mathbf{R} \right\}$$

Example 1. $\mathcal{V} = \text{span}\,\{v_1, v_2\} = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \subset \mathbf{R}^3.$

- The set of vectors $v_1, \ldots, v_r$ is called *linearly independent* if $\sum_{k=1}^{r} \alpha_k v_k = 0$ implies $\alpha_k = 0$, $k = 1, \ldots, r$. A linearly independent set of vectors that spans $\mathcal{V}$ is called a *basis* for $\mathcal{V}$.

  If $\mathcal{V} = \text{span}\{v_1, \ldots, v_r\}$ where the vectors v_k, $k = 1, \ldots, r$ are linearly independent (i.e. a basis for $\mathcal{V}$) then we say that $\mathcal{V}$ has dimension r, which is denoted $\dim \mathcal{V} = r$.

- Subspace addition is defined as

$$\mathcal{V}_1 + \mathcal{V}_2 = \{v_1 + v_2 : v_1 \in \mathcal{V}_1, v_2 \in \mathcal{V}_2\}$$

  If the vectors in $\mathcal{V}_1$ and $\mathcal{V}_2$ are linearly independent the we write the vector space sum is a direct sum $\mathcal{V}_1 \oplus \mathcal{V}_2$.

- If $\mathbf{R}^n = \mathcal{V}_1 \oplus \mathcal{V}_2$ then $\mathcal{V}_2 = \mathbf{R}^n \ominus \mathcal{V}_1$.

- If $\mathcal{V}_1 = \text{span}\{v_1, \ldots, v_4\}$ and $\mathcal{V}_2 = \text{span}\{v_4, \ldots, v_5\}$, where $v_1, \ldots, v_5$ are linearly independent. Then $\mathcal{V}_1 + \mathcal{V}_2 = \text{span}\{v_1, \ldots, v_5\}$.

- If $\mathcal{V}_1 = \text{span}\{v_1, \ldots, v_4\}$ and $\mathcal{V}_2 = \text{span}\{v_5\}$, where $v_1, \ldots, v_5$ are linearly independent. Then

$$\mathcal{V}_1 + \mathcal{V}_2 = \mathcal{V}_1 \oplus \mathcal{V}_2 = \text{span}\{v_1, \ldots, v_5\}$$

  In other words, in this example the subspace addition is a direct sum.

- A subspace $\mathcal{V}$ is A-invariant if $A\mathcal{V} \subset \mathcal{V}$. This means that for any $v \in \mathcal{V}$, $Av \in \mathcal{V}$.

Cayley Hamilton Theorem

Theorem 1: Any square matrix A satisfies its characteristic polynomial. In other words, given the characteristic polynomial of the matrix $A \in \mathbf{R}^{n \times n}$

$$\chi_A(\lambda) = \det(\lambda I - A) = \lambda^n + a_1 \lambda^{n-1} + \ldots + a_n$$

then

$$\chi(A) = A^n + a_1 A^{n-1} + \ldots + a_n I = 0$$

Remark 1: The theorem has two important consequences:

1 $A^n = -a_1 A^{n-1} - a_2 A^{n-2} - \ldots - \alpha_n I$

2 Every matrix polynomial $\psi(A)$ of order $n + i, i \geq 0$ can be expressed by a $(n-1)$-order polynomial. Note that e^{At} is an infinity order polynomial which can also be expressed as a $(n-1)$-order polynomial.

In particular, in the lecture we use that

$$A^k = \sum_{l=0}^{n-1} \alpha_{kl} A^l, \qquad \forall k \geq 0$$

Note if $k < n$ then

$$\alpha_{kl} = \begin{cases} 1, & l = k \\ 0, & \text{otherwise} \end{cases}$$

Reachability for Time-invariant Systems

$$\dot{x}(t) = Ax(t) + Bu(t)$$

Definition 1. *The controllability Gramian for a time-invariant system becomes*

$$W(t_0, t_1) = \int_0^{t_1 - t_0} e^{At} B B^T e^{A^T t} dt$$

Definition 2. *The controllability matrix is defined as*

$$\Gamma = \begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix}$$

Theorem 1. $\text{Im}\,(W(t_0, t_1)) = \text{Im}\,(\Gamma)$

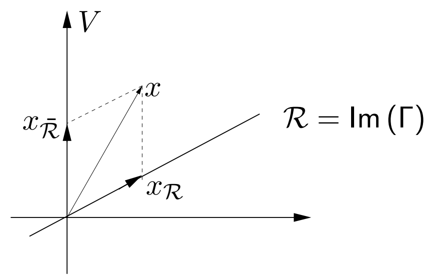
Definition 3. *The pair (A, B) is called completely reachable if* $\text{rank}(\Gamma) = \dim \text{Im}\,(\Gamma) = n$.

Definition 4. *The reachable subspace is defined as* $\mathcal{R} = \text{Im}\,(\Gamma) = \{x \in \mathbf{R}^n : x \text{ can be reached from the origin}\}$.

Theorem 2. *The reachable subspace is A-invariant, i.e. $A\mathcal{R} \subset \mathcal{R}$.*

Corollary 1. $e^{At}\mathcal{R} \subset \mathcal{R}$.

Decomposition



- $\mathcal{R} = \text{Im}\,\Gamma$ the reachable subspace.
- $\mathbf{R}^n = \text{Im}\,(\Gamma) \oplus V$, $\dim(V) = n - r$, where $r = \dim \text{Im}\,(\Gamma)$
- Consider new state vector with components
  - $x_{\mathcal{R}} \in \mathcal{R}$ the reachable part of the state vector
  - $x_{\bar{\mathcal{R}}} \in V$ states that cannot be reached from the origin

Dynamics in new coordinates

$$\begin{bmatrix} \dot{x}_{\mathcal{R}} \\ \dot{x}_{\bar{\mathcal{R}}} \end{bmatrix} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{bmatrix} \begin{bmatrix} x_{\mathcal{R}} \\ x_{\bar{\mathcal{R}}} \end{bmatrix} + \begin{bmatrix} \tilde{B}_1 \\ 0 \end{bmatrix} u$$

The reachability matrix

$$\Gamma = \begin{bmatrix} \tilde{B}_1 & \tilde{A}_{11}\tilde{B}_1 & \dots & \tilde{A}_{11}^{n-1}\tilde{B}_1 \\ 0 & 0 & \dots & 0 \end{bmatrix} \sim \begin{bmatrix} \tilde{\Gamma} & 0 \\ 0 & 0 \end{bmatrix}$$

where $\tilde{\Gamma} = \begin{bmatrix} \tilde{B}_1 & \tilde{A}_{11}\tilde{B}_1 & \dots & \tilde{A}_{11}^{r-1}\tilde{B}_1 \end{bmatrix}$ The last equivalence is a consequence of the Cayley-Hamilton theorem. $\tilde{A}_{11}$ is $r \times r$ so the last $n-r$ vectors in Γ are linear combinations of the vectors in $\tilde{\Gamma}$.

Equivalent Matrices

- Two matrices A and B are called equivalent if $A = PBQ$, for some invertible matrices P, Q of suitable size.
- The matrices P and Q normally represents elementary row respectively column operations.
- Above we used the notation $A \sim B$ to indicate that A and B are equivalent.

Example 2. *We have*

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Distinction Between Time-varying and Time-invariant Systems

The state $x(t_1) = x_1$ can be reached from $x(t_0) = 0$ if

- $x_1 \in \text{Im}\, W(t_0, t_1)$ (when the system is linear time-varying (LTV))
  - Since $W(t_0, t_2) \geq W(t_0, t_1)$ when $t_2 > t_1$ it follows that the chances of reaching x_1 from 0 increases the longer we try.
- $x_1 \in \mathcal{R} = \text{Im}\, \Gamma$ (when the system is linear time-invariant (LTI))
  - Note that this condition is independent of the length of the time-interval. If we can steer from 0 to x_1 then we can do it arbitrarily fast.

The situation is more complicated if we start from arbitrary $x(t_0)$. Why?