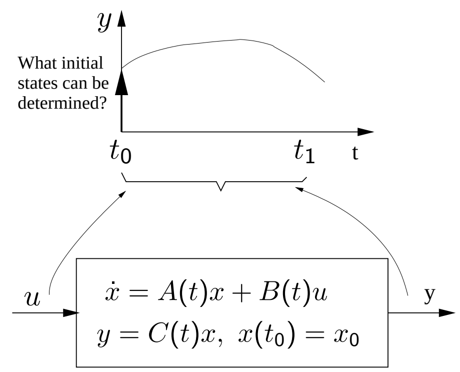
Observability

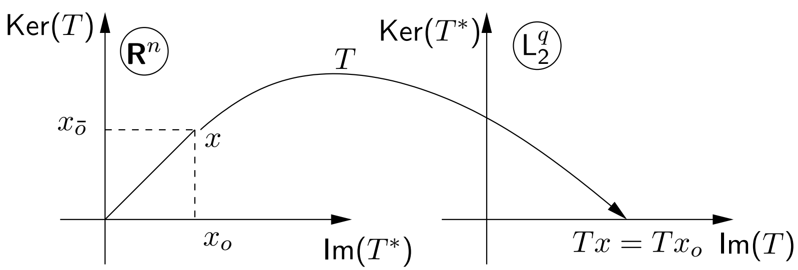
1. The observability problem
2. Observability for time-invariant systems.
3. Decomposition of the state space.
4. When is the water tank system completely reachable?

The Observability Problem



- Use inputs and outputs to determine the initial condition
- Mathematical formulation: Find $x_0 \in \mathbf{R}^n$ such that

$$Tx_0 = v, \quad \text{where} \quad \begin{cases} (Tx_0)(t) = C(t)\Phi(t,t_0)x_0 \\ v(t) = y(t) - \int_{t_0}^{t} \Phi(t,\tau)Bu(\tau)d\tau \end{cases}$$



- $x_{\bar{o}} \in$ Ker (T) can not be observed in output
- Im $(T) \subset L_2^q[t_0, t_1]$ (function space)
- Ker $(T) =$ Ker (T^*T) since Ker $(T^*) =$ Im $(T)^\perp$
  - Ker (T^*T) is easy to compute since $T^*T \in \mathbf{R}^{n\times n}$
- $M = T^*T$ is called the observability Gramian

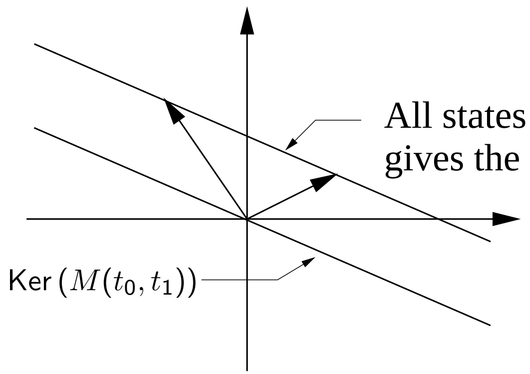
- $T^* : L_2^q[0,\infty) \to \mathbf{R}^n$ is defined by

$$T^*v = \int_{t_0}^{t_1} \Phi(t,t_0)^T C(t)^T v(t)dt$$

and the observability Gramian becomes

$$M(t_0,t_1) = T^*T = \int_{t_0}^{t_1} \Phi(t,t_0)^T C(t)^T C(t)\Phi(t,t_0)dt$$

Given $u(t)$, $t \in [t_0, t_1]$. Then $x(t_0) = x_k$, $k = 1, 2$, gives the same output if and only if $x_2 - x_1 \in \text{Ker}\,(M(t_0, t_1))$.



Observability for Time-invariant Systems

$$\dot{x}(t) = Ax(t), \qquad x(0) = x_0$$
$$y(t) = Cx(t)$$

Definition 1. *The Observability Gramian for a time-invariant system is*

$$M(t_0, t_1) = \int_0^{t_1 - t_0} e^{A^T t} C^T C e^{At} dt$$

Definition 2. *The observability matrix is defined as*

$$\Omega = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

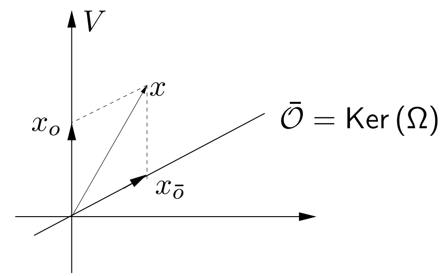
Theorem 1. $\text{Ker}\,(M(t_0, t_1)) = \text{Ker}\,(\Omega)$

Definition 3. *The pair (C, A) is called completely observable if* $\text{rank}(\Omega) = n$.

Definition 4. *The unobservable subspace is defined as $\bar{\mathcal{O}} = \text{Ker}\,\Omega$.*

Theorem 2. *The unobservable subspace is A-invariant, i.e., $A\bar{\mathcal{O}} \subset \bar{\mathcal{O}}$.*

Decomposition of state space



- $\bar{\mathcal{O}} = \text{Ker}\,(T)$ unobservable states.
- $\mathbf{R}^n = \text{Ker}\,(\Omega) \oplus V$, $\dim(V) = n - r$, where $r = \dim \text{Ker}\,(\Omega)$
- Consider new state vector with components
  - $x_o \in V$ the observable part of the state vector.
  - $x_{\bar{o}} \in \text{Ker}\,(\Omega)$ unobservable states

Dynamics in new coordinates

$$\begin{bmatrix} \dot{x}_o \\ \dot{x}_{\bar{o}} \end{bmatrix} = \begin{bmatrix} \tilde{A}_{11} & 0 \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix} \begin{bmatrix} x_o \\ x_{\bar{o}} \end{bmatrix}$$

$$y = \begin{bmatrix} \tilde{C}_1 & 0 \end{bmatrix} \begin{bmatrix} x_o \\ x_{\bar{o}} \end{bmatrix}$$

Observability matrix

$$\Omega = \begin{bmatrix} \tilde{C}_1 & 0 \\ \tilde{A}_{11}\tilde{C}_1 & 0 \\ \vdots & \\ \tilde{A}_{11}^{n-1}\tilde{C}_1 & 0 \end{bmatrix} \sim \begin{bmatrix} \tilde{\Omega} & 0 \\ 0 & 0 \end{bmatrix}, \quad \tilde{\Omega} = \begin{bmatrix} \tilde{C}_1 \\ \tilde{A}_{11}\tilde{C}_1 \\ \vdots \\ \tilde{A}_{11}^{r-1}\tilde{C}_1 \end{bmatrix}$$

The last equivalence is a consequence of the Cayley-Hamilton theorem. $\tilde{A}_{11}$ is $r \times r$ so the last $n-r$ rows in Ω are linear combinations of the rows in $\tilde{\Omega}$.

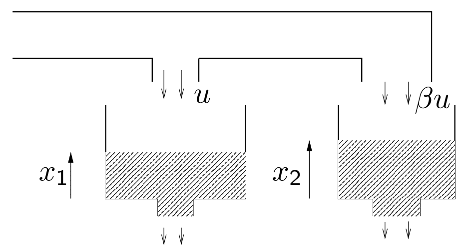
- Above we used the notation $A \sim B$ to indicate that A and B are equivalent.
- Two matrices A and B are called equivalent if $A = PBQ$, for some invertible matrices P, Q of suitable size.
- The matrices P and Q normally represents elementary row respectively column operations.

Distinction Between Time-varying and Time-invariant Systems

The states $x_1, x_2 \in \mathbf{R}^n$ cannot be distinguished by observing the output over the time interval $[t_0, t_1]$ if

- $x_2 - x_1 \in \operatorname{Ker} M(t_0, t_1)$ (when the system is linear time-varying (LTV))
  - Since $M(t_0, t_2) \geq M(t_0, t_1)$ when $t_2 \geq t_1$ it follows that the chances of distinguishing x_2 from x_1 increases the longer we observe the output.
- $x_2 - x_1 \in \operatorname{Ker} \Omega$ (when the system is linear time-invariant (LTI))
  - This condition is independent of time. If we can't distinguish x_2 from x_1 by observing the output over $[t_0, t_1]$ then it will not help if we wait longer.

When is the Water Tank System Completely Reachable



Linearized dynamics

$$\dot{x}_1 = -x_1 + u$$
$$\dot{x}_2 = -\alpha x_2 + \beta u$$

Since it is an time-invariant system we may use that $\operatorname{Im} W(t_0, t_1) = \operatorname{Im} \Gamma$.

We consider two cases

1. If $\alpha = 1$ then

$$\operatorname{Im}\Gamma = \operatorname{Im}\begin{bmatrix} 1 & -1 \\ \beta & -\beta \end{bmatrix} = \operatorname{span}\left\{\begin{bmatrix} 1 \\ \beta \end{bmatrix}\right\}$$

   Constant relation between the two water levels.

2. If $\beta = 1$ then

$$\operatorname{Im}\Gamma = \operatorname{Im}\begin{bmatrix} 1 & -1 \\ 1 & -\alpha \end{bmatrix} = \begin{cases} \operatorname{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}, & \alpha = 1 \\ \operatorname{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ \alpha \end{bmatrix}\right\}, & \alpha \neq 1 \end{cases}$$

   Any desired water levels are achievable if $\alpha = 1$.