Realization Theory, Part I

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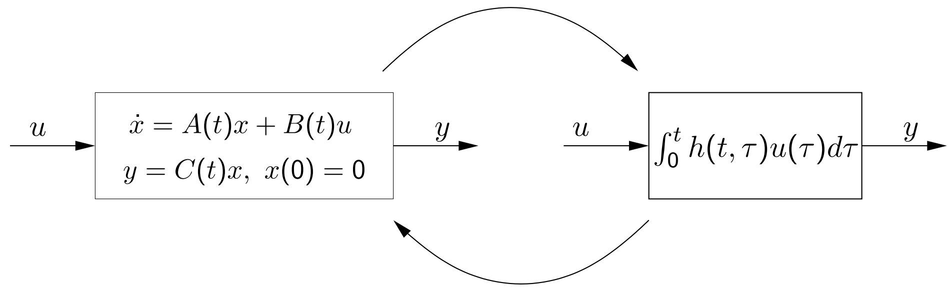
The Realization Problem

$$h(t,\tau) = C(t)\Phi(t,\tau)B(\tau)$$



The realization problem

Theorem 1. *The realization problem can be solved if and only if $h(t,\tau) = H(t)F(\tau)$ for some functions $H(t)$ and $F(t)$ with appropriate dimensions.*

Proof of necessity: We know from chapter 2 that

$$h(t,\tau) = C(t)\Phi(t,\tau)B(\tau) = \underbrace{C(t)\Phi(t,0)}_{H(t)}\underbrace{\Phi(0,\tau)B(\tau)}_{F(\tau)}$$

Proof of sufficiency: Consider

$$\dot{x}(t) = F(t)u(t), \quad x(t_0) = 0$$
$$y(t) = H(t)x(t)$$

This implies

$$y(t) = H(t)x(t) = H(t)\int_{t_0}^{t} F(\tau)u(\tau)d\tau = \int_{t_0}^{t} \underbrace{H(t)F(\tau)}_{h(t,\tau)} u(\tau)d\tau$$

which is the desired convolution kernel.

Time-invariant Systems

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = 0$$
$$y(t) = C(x(t) + Du(t)$$
$$\Leftrightarrow \quad y(t) = \int_0^t g(t-\tau)u(\tau)d\tau, \qquad g(t) = Ce^{A(t-\tau)}B\theta(t) + D\delta(t)$$
$$\Leftrightarrow \quad Y(s) = G(s)U(s), \qquad G(s) = \mathcal{L}\{g\}(s) = C(sI-A)^{-1}B + D$$

The transfer function $G(s) = C(sI-A)^{-1}B + D$ is

- a rational function in s
- a proper function since $G(\infty) = D$ and strictly proper if $D = 0$

The Fundamental Problem

Given a matrix function with proper rational elements

$$G(s) = \begin{bmatrix} G_{11}(s) & \dots & G_{1m}(s) \\ \vdots & & \vdots \\ G_{q1}(s) & \dots & G_{qm}(s) \end{bmatrix}, \qquad G_{ij}(s) = \frac{n_{ij}(s)}{d_{ij}(s)}$$

where $d_{ij}(s), n_{ij}(s)$ are polynomials with $\deg(n_{ij}) \le \deg(d_{ij})$. Find a finite dimensional realization, i.e. matrices $(A, B, C, D) \in \mathbf{R}^{n\times n} \times \mathbf{R}^{n\times m} \times \mathbf{R}^{q\times n} \times \mathbf{R}^{q\times m}$ such that

$$G(s) = C(sI - A)^{-1}B + D$$

- What is the minimal dimension of the realization, i.e. the smallest possible $n = \dim(A)$.
- Is the minimal realization unique?

The direct term is obtained as $D = G(\infty)$. The main problem is to find (A, B, C) such that

$$R(s) = G(s) - G(\infty) = C(sI - A)^{-1}B$$

Standard Reachable Realization

Example 1.

$$Y(s) = \frac{n_2s^2 + n_1s + n_0}{s^3 + a_1s^2 + a_2s + a_3}U(s)$$

$$\Leftrightarrow \quad \frac{1}{n_2s^2 + n_1s + n_0}Y(s) = \underbrace{\frac{1}{s^3 + a_1s^2 + a_2s + a_3}U(s)}_{X_1(s)}$$

Let $X_2(s) = sX_1(s)$ *and* $X_3(s) = sX_2(s)$. *Then*

$$sX_3(s) = -a_1X_2(s) - a_2X_1(s) - a_3X_1(s) + U(s)$$
$$Y(s) = n_0X_1(s) + n_1X_2(s) + n_2X_3(s)$$

Inverse Laplace transformation of the above equations gives the realization

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} n_0 & n_1 & n_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Multivariable Case

$$R(s) = \frac{1}{\chi(s)}\left(N_0 + N_1 s + \ldots + N_{r-1}s^{r-1}\right)$$
$$\chi(s) = s^r + a_1 s^{r-1} + \ldots + a_r, \quad \text{(Least common denominator)}$$

The standard reachable form:

$$A = \begin{bmatrix} 0 & I & 0 & \ldots & 0 \\ 0 & 0 & I & \ldots & 0 \\ \ldots & & & \ddots & \\ \ldots & & & & I \\ -a_r I & -a_{r-1}I & -a_{r-2}I & \ldots & -a_1 I \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ I \end{bmatrix}$$

$$C = \begin{bmatrix} N_0 & N_1 & N_2 & \ldots N_{r-2} & N_{r-1} \end{bmatrix}$$

The standard reachable realization is completely reachable since

$$\Gamma = \begin{bmatrix} B & AB & \ldots & A^{n-1}B \end{bmatrix} = \begin{bmatrix} 0 & 0 & \ldots & 0 & I \\ 0 & 0 & \ldots & I & * \\ & & \cdot^{\cdot^{\cdot}} & & \\ 0 & I & \ldots & * & * \\ I & * & \ldots & * & * \end{bmatrix}$$

which has full rank.

Standard Observable Form

Example 2.

$$R(s) = \frac{n_2 s^2 + n_1 s + n_0}{s^3 + a_1 s^2 + a_2 s + a_3} = r_1 s^{-1} + r_2 s^{-2} + r_3 s^{-3} \ldots$$

The Markov parameters r_k can easily be shown to satisfy the recursion

$$\begin{aligned} r_1 &= n_2 \\ r_2 &= -a_1 r_1 + n_1 \\ r_3 &= -a_1 r_2 - a_2 r_1 + n_0 \\ r_{3+i} &= -a_1 r_{2+i} - a_2 r_{1+i} - a_3 r_i \end{aligned} \qquad (1)$$

We must have

$$R(s) = C(sI - A)^{-1}B = CBs^{-1} + CABs^{-2} + CA^2Bs^{-3} \ldots$$

It is easy to see that

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix}, \qquad B = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

satisfies $r_k = CA^{k-1}B$. The proof follows since

$$A^{k-1}B = \begin{bmatrix} r_k \\ r_{k+1} \\ r_{k+2} \end{bmatrix}$$

which can be proven from (1) by induction.

Multi-variable Case

$R(s) = R_1 s^{-1} + R_2 s^{-2} + R_3 s^{-3} + \dots$ where R_k is k^{th} Markov parameter

The standard observable realization

$$A = \begin{bmatrix} 0 & I & 0 & \dots & 0 \\ 0 & 0 & I & \dots & 0 \\ \dots & & & \ddots & \\ \dots & & & & 1 \\ -a_r I & -a_{r-1} I & -a_{r-2} I & \dots & -a_1 I \end{bmatrix}, \qquad B = \begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_{r-1} \\ R_r \end{bmatrix}$$

$$C = \begin{bmatrix} I & 0 & 0 & \dots & 0 & 0 \end{bmatrix}$$

The standard observable realization is completely observable since

$$\Omega = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-2} \\ CA^{n-1} \end{bmatrix} = \begin{bmatrix} I & 0 & 0 & \dots & 0 & 0 \\ 0 & I & 0 & \dots & 0 & 0 \\ & & & \ddots & & \\ 0 & 0 & 0 & \dots & I & 0 \\ 0 & 0 & 0 & \dots & 0 & I \end{bmatrix}$$

Example

$$R(s) = \begin{bmatrix} \frac{1}{s^2+4s+3} & \frac{1}{s+3} \end{bmatrix} = \frac{1}{s^2+4s+3} \left(\begin{bmatrix} 1 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 1 \end{bmatrix} s \right)$$

which gives the standard reachable realization (controllable canonical form)

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -3 & 0 & -4 & 0 \\ 0 & -3 & 0 & -4 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 3 & 0 & 1 \end{bmatrix}$$

Is it observable? We have

$$\Omega = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & -3 & 1 & -1 \\ -3 & 3 & -4 & 1 \\ 12 & -3 & 13 & -1 \end{bmatrix}$$

We have the equivalence $\Omega Q = \Omega_1$, where

$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & -3 & 0 & 1 \end{bmatrix}, \qquad \Omega_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -4 & 1 \\ 0 & 0 & 13 & -1 \end{bmatrix}$$

Hence

$$\bar{\mathcal{O}} = \text{Ker}\,\Omega = Q\text{Ker}\,\Omega_1 = \text{span}\left\{\begin{bmatrix}1\\0\\-1\\-1\end{bmatrix}, \begin{bmatrix}0\\1\\0\\-3\end{bmatrix}\right\}$$

Alternatively

$$R(s) = \begin{bmatrix}\frac{1}{s^2+4s+3} & \frac{1}{s+3}\end{bmatrix} = s^{-1}\begin{bmatrix}1 & 0\end{bmatrix} + s^{-2}\begin{bmatrix}1 & -1\end{bmatrix} + s^{-3}\begin{bmatrix}-4 & 1\end{bmatrix}\dots$$

which gives the standard observable form (observable canonical form)

$$A = \begin{bmatrix}0 & 1\\-3 & -4\end{bmatrix}, \quad B = \begin{bmatrix}0 & 1\\1 & -1\end{bmatrix}$$

$$C = \begin{bmatrix}1 & 0\end{bmatrix}$$

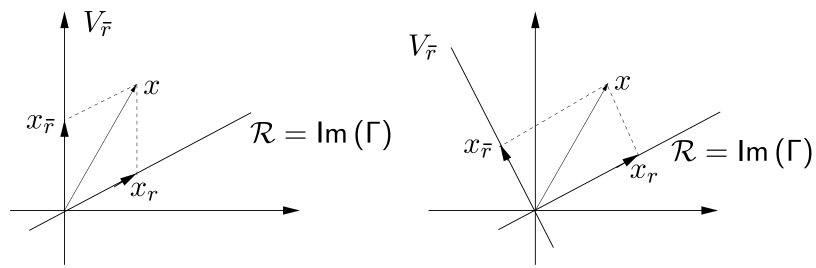
This is a controllable realization

$$\Gamma = \begin{bmatrix}B & AB\end{bmatrix} = \begin{bmatrix}0 & 1 & 1 & -1\\1 & -1 & -4 & 1\end{bmatrix}$$

Reducing the Order

- The standard reachable realization is reachable but possibly not observable
  - If not obserable then there are states to be removed.
- The standard observable realization is observable but possibly not reachable
  - If not reachable then there are states to be removed.

Reachability



The choice of $V_{\bar{r}}$ is not unique

- $\mathcal{R} = \text{Im}\,\Gamma = \text{Im}\begin{bmatrix}B & AB & \dots & A^{n-1}B\end{bmatrix}$ is the reachable subspace.
- The reachable subspace is A-invariant, i.e. $A\mathcal{R} \subseteq \mathcal{R}$

  $\Rightarrow \mathcal{R}$ is independent of the choice of coordinates
- $\mathbf{R}^n = \text{Im}\,\Gamma \oplus V_{\bar{r}}$, dim $(V_{\bar{r}}) = n - r$, where $r = \text{dim Im}\,\Gamma$
  - $V_{\bar{r}}$ is not unique. It depends on the choice of coordinates.

The dynamics in the new coordinates

- $x_r \in \mathcal{R}$ reachable states
- $x_{\bar{r}} \in V_{\bar{r}}$ states that cannot be reached from the origin

is on the following form (independent of the choice of $V_{\bar{r}}$)

$$\begin{bmatrix} \dot{x}_r \\ \dot{x}_{\bar{r}} \end{bmatrix} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{bmatrix} \begin{bmatrix} x_r \\ x_{\bar{r}} \end{bmatrix} + \begin{bmatrix} \tilde{B}_1 \\ 0 \end{bmatrix} u \qquad \begin{cases} \tilde{A}_{21} = 0 \text{ since } \tilde{A}\mathcal{R} \subseteq \mathcal{R} \\ \tilde{B}_2 = 0 \text{ since } \mathrm{Im}\, \tilde{B} \subseteq \mathcal{R} \end{cases}$$

$$y = \begin{bmatrix} \tilde{C}_1 & \tilde{C}_2 \end{bmatrix} \begin{bmatrix} x_r \\ x_{\bar{r}} \end{bmatrix}$$

Reduced order dynamics:

$$R(s) = \tilde{C}_1(sI - \tilde{A}_{11})^{-1}\tilde{B}_1$$

Observability

- $\bar{\mathcal{O}} = \mathrm{Ker}\,\Omega = \mathrm{Ker} \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$ is the unobservable subspace.
- The unobservable subspace is A-invariant, i.e. $A\bar{\mathcal{O}} \subseteq \bar{\mathcal{O}}$

  $\Rightarrow \bar{\mathcal{O}}$ is independent of the choice of coordinates.
- $\mathbf{R}^n = \bar{\mathcal{O}} \oplus V_o$
  - V_o is not unique. It depeneds on the choice of coordinates.

The dynamics in new coordinates

- $x_{\bar{o}} \in \bar{\mathcal{O}} = \mathrm{Ker}\,(\Omega)$ unobservable states
- $x_o \in V_o$ states that can be observed.

is on the following form (no matter the choice of V_o)

$$\begin{bmatrix} \dot{x}_{\bar{o}} \\ \dot{x}_o \end{bmatrix} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{bmatrix} \begin{bmatrix} x_{\bar{o}} \\ x_o \end{bmatrix} + \begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{bmatrix} u \qquad \begin{cases} \tilde{A}_{21} = 0 \text{ Since } A\bar{\mathcal{O}} \subseteq \bar{\mathcal{O}} \\ \tilde{C}_1 = 0 \text{ since } \bar{\mathcal{O}} \subseteq \mathrm{Ker}\,(\tilde{C}) \end{cases}$$

$$y = \begin{bmatrix} 0 & \tilde{C}_2 \end{bmatrix} \begin{bmatrix} x_{\bar{o}} \\ x_o \end{bmatrix}$$

Reduced order dynamics:

$$R(s) = \tilde{C}_2(sI - \tilde{A}_{22})^{-1}\tilde{B}_2$$

- The Kalman decomposition generalizes the above ideas
- A state-space realization is not minimal if it there are unreachable or unobservable states.
- Answers to the fundamental problem are given next time.