Realization Theory, Part II

1. The Kalman decomposition
2. Minimal realizations
3. Hankel matrices and characterization of minimal realizations
4. Overview of theory

The Kalman Decomposition

$\mathcal{R} \cap \bar{\mathcal{O}} = \text{span}\{V_{\bar{o}r}\}$	$\mathcal{V}_{or} = \text{span}\{V_{or}\}$
$\mathcal{V}_{\bar{o}\bar{r}} = \text{span}\{V_{\bar{o}\bar{r}}\}$	$\mathcal{V}_{o\bar{r}} = \text{span}\{V_{o\bar{r}}\}$

A change of basis

$$x = \begin{bmatrix} V_{\bar{o}r} & V_{or} & V_{\bar{o}\bar{r}} & V_{o\bar{r}} \end{bmatrix} \begin{bmatrix} z_{\bar{o}r} \\ z_{or} \\ z_{\bar{o}\bar{r}} \\ z_{o\bar{r}} \end{bmatrix}$$

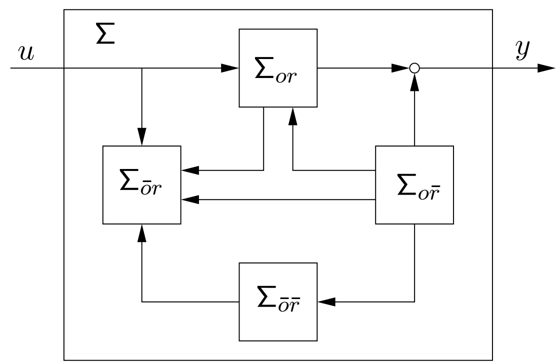
The Kalman Decomposition

gives

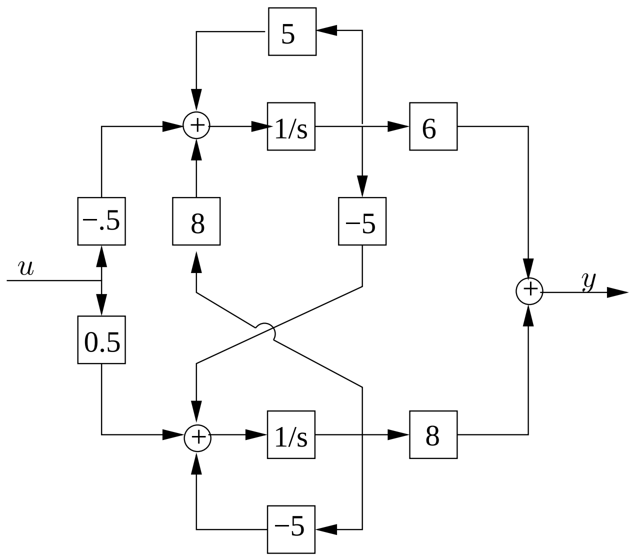
$$\begin{bmatrix} \dot{z}_{\bar{o}r} \\ \dot{z}_{or} \\ \dot{z}_{\bar{o}\bar{r}} \\ \dot{z}_{o\bar{r}} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ 0 & A_{22} & 0 & A_{24} \\ 0 & 0 & A_{33} & A_{34} \\ 0 & 0 & 0 & A_{44} \end{bmatrix} \begin{bmatrix} z_{\bar{o}r} \\ z_{or} \\ z_{\bar{o}\bar{r}} \\ z_{o\bar{r}} \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & C_2 & 0 & C_4 \end{bmatrix} \begin{bmatrix} z_{\bar{o}r} \\ z_{or} \\ z_{\bar{o}\bar{r}} \\ z_{o\bar{r}} \end{bmatrix}$$

Kalman Decomposition



Example



Dynamics

$$\dot{x} = \begin{bmatrix} 5 & 8 \\ -3 & -5 \end{bmatrix} x + \frac{1}{2}\begin{bmatrix} -1 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 6 & 8 \end{bmatrix} x$$

We have

$$\mathcal{R} = \text{Im}\,\Gamma = \mathbf{R}^2 \qquad \text{and} \qquad \bar{\mathcal{O}} = \text{Ker}\,\Omega = \text{span}\left\{\begin{bmatrix} -4 \\ 3 \end{bmatrix}\right\}$$

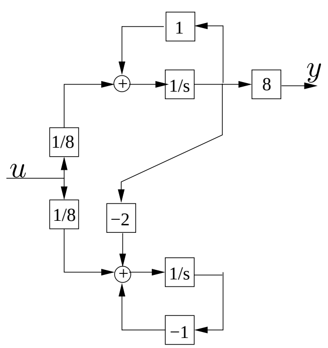
A possible complementary space such that $\mathbf{R}^2 = \bar{\mathcal{O}} \oplus V_0$ is

$$V_0 = \text{span}\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$$

In the new coordinates

$$\begin{bmatrix} \dot{z}_{\bar{o}} \\ \dot{z}_o \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} z_{\bar{o}} \\ z_o \end{bmatrix} + \frac{1}{8}\begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 8 \end{bmatrix} \begin{bmatrix} z_{\bar{o}} \\ z_o \end{bmatrix}$$



Minimal Realizations

Let $R(s)$ be a strictly proper rational transfer function.

Definition 1. *The McMillan degree, $\delta(R)$, is the dimension of any minimal realization (A, B, C) such that $R(s) = C(sI - A)^{-1}B$. This means that if $(\tilde{A}, \tilde{B}, \tilde{C})$ is another realization then* dim $(A) \leq$ dim $(\tilde{A})$.

We will

- Identify necessary and sufficient conditions for a realization to be minimal.
- Show how the McMillan degree can be computed using the Markov parameters
- Show how to obtain a minimal realization.

The Hankel Matrix

Given a strictly proper rational transfer matrix $R(s)$ such that

$$\begin{aligned} R(s) &= \frac{1}{\chi(s)}\left(N_0 + N_1 s + \dots N_{r-1}s^{r-1}\right) \\ &= R_1 s^{-1} + R_2 s^{-2} + R_3 s^{-3} \dots \end{aligned}$$

$\chi(s) = s^r + a_1 s^{r-1} + \dots + a_r$ (is the least common denominator of the elements $R_{ij}(s)$ of $R(s)$)

the block Hankel matrix is defined from the Markov parameters as

$$H_\nu = \begin{bmatrix} R_1 & R_2 & \dots & R_\nu \\ R_2 & R_3 & \dots & R_{\nu+1} \\ \vdots & & & \vdots \\ R_\nu & R_{\nu+1} & \dots & R_{2\nu-1} \end{bmatrix}$$

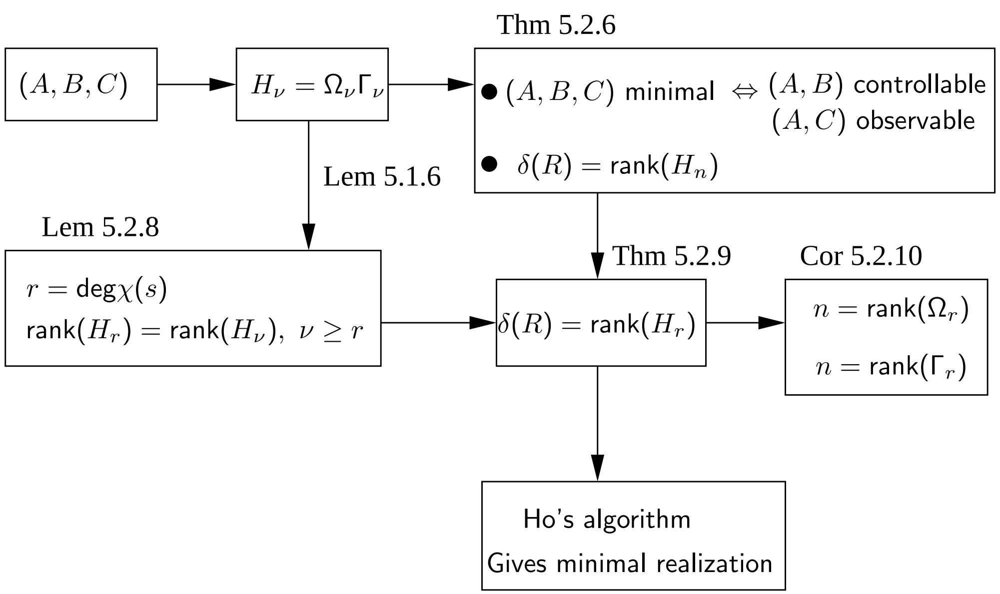
Given any realization (A, B, C), the Markov parameters are given as $R_k = CA^{k-1}B$. This implies that

$$H_r = \begin{bmatrix} CB & CAB & \dots & CA^{\nu-1}B \\ CAB & CA^2B & \dots & CA^\nu B \\ \vdots & & \dots & \vdots \\ CA^{\nu-1}B & CA^\nu B & \dots & CA^{2\nu-2}B \end{bmatrix}$$

$$= \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{\nu-1} \end{bmatrix} \begin{bmatrix} B & AB & \dots & A^{\nu-1}B \end{bmatrix} = \Omega_\nu \Gamma_\nu$$

If $(\tilde{A}, \tilde{B}, \tilde{C})$ is another realization then $H_\nu = \tilde{\Omega}_\nu \tilde{\Gamma}_\nu = \Omega_\nu \Gamma_\nu$

Overview of Theory



Theorem 5.2.6

Theorem 1. *A realization is minimal if and only if*

- (A, B) *is completely reachable.*
- (A, C) *is completely observable.*

The McMillan degree is given as $\delta(R) = \text{rank}(H_n)$.

Proof. See Lindquist and Sand or the lecture. The proof relies on properties of the matrix rank presented in the next few slides. □

Matrix Rank

Definition 2. *The rank of $P \in \mathbf{R}^{q\times m}$ is defined as* rank $P :=$ dim (Im P).

Lemma 1. *We have*

(i) rank $(PQ) \leq$ rank (P)

(ii) rank $(PQ) \leq$ rank (Q)

(iii) rank $(P) =$ rank (P^T)

(iv) rank $(QQ^T) =$ rank (Q)

Proof of Lemma

(i) Each column in PQ is a linear combination of the columns in P. This implies Im $(PQ) \subset$ Im $(P) =$ span$\{p_1, \ldots, p_m\}$, where p_k are the columns of P. This implies rank $(PQ) \leq$ rank (P).

(iii) The fundamental theorem of linear algebra shows that dim (Im (P)) $=$ dim (Im (P^T)). This proves (iii).

(ii) rank $(PQ) \overset{(iii)}{=}$ rank $(Q^TP^T) \overset{(i)}{\leq}$ rank $(Q^T) \overset{(iii)}{=}$ rank (Q)

(iv) The fundamental theorem of linear algebra shows that Im $(QQ^T) =$ Im (Q), which implies rank $(QQ^T) =$ rank (Q).

Equivalent Matrices

- Two matrices A and B are called equivalent if $A = PBQ$, for some invertible matrices P, Q of suitable size.
- The matrices P and Q normally represents elementary row respectively column operations.
- We use the notation $A \sim B$ to indicate that A and B are equivalent.

Example 1. *We have*

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

We have shown $PHQ = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, where

$$P = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}, \quad H = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Lemma 5.2.8.

Proposition 1. *Let* $r = \deg(\chi(s))$. *Then* $\text{rank}(H_r) = \text{rank}(H_\nu)$, $\nu \geq r$.

Proof. Use $R_{r+k} = -a_1 R_{r+k-1} - \ldots - a_r R_k$ to show that H_r and H_ν are equivalent. □

The following characterization of the natrix inverse is used for the proof of Lemma 5.2.9.

Lemma 2. $(sI - A)^{-1} = \frac{1}{\det(sI - A)} \text{adj}\,(sI - A)$ *where the* (i, j) *element of* $\text{adj}\,(sI - A)$ *is* $(-1)^{i+j}$ *times the determinant of the submatrix obtained by removing the* j^{th} *row and* i^{th} *column of* $(sI - A)$. *If* A *is* $n \times n$ *then* $\deg(\text{adj}\,(sI - A)) < n$.

Theorem 5.2.9

Theorem 2. $\delta(R) = \text{rank}\,(H_r)$, *where* $r = \deg(\chi(s))$.

Proof. Theorem 5.2.6 shows $n := \delta(R) = \text{rank}\,(H_n)$. Suppose (A, B, C) is a minimal realization. We have

$$
\begin{aligned}
R(s) &= \frac{1}{\chi(s)} \left(N_0 + N_1 s \ldots N_{r-1} s^{r-1}\right) = C(sI - A)^{-1} B \\
&= \frac{1}{\chi_A(s)} C(\text{adj}(sI - A))B, \quad \text{where} \quad \chi_A(s) = \det(sI - A)
\end{aligned}
$$

From this it is clear that $r = \deg(\chi(s)) \leq \deg(\chi_A(s)) = n$. By the minimality of (A, B, C) we have

$$\text{rank}\,(H_r) = \text{rank}\,(H_n) = \delta(R) \;\Rightarrow\; \delta(R) = \text{rank}\,(H_r)$$

□