SF2832 - Mathematical systems theory, Autumn 2016
Exercise session 1 - Computing the matrix exponential

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For a linear time-invariant dynamical system

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1}$$
$$x(0) \text{ given,}$$

the solution is given by

$$x(t) = e^{At}x(0) + \int_0^t e^{A\tau}Bu(\tau)d\tau. \tag{2}$$

The matrix exponential e^{At} is thus fundamental in describing such systems. It is defined as

$$e^{At} := \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k.$$

There are (at least) three different ways to compute the matrix exponential:

i) using the definition,

ii) using the Laplace transform,

iii) diagonalization or Jordan form.

i) computing it using the definition. As we saw in first exercise session, this approach is in general only possible in two special cases: either if the matrix is nilpotent, i.e., if $A^k = 0$ for some finite value of k (for computational tractability, k needs to be a relatively small number), or if A is a diagonal matrix.

ii) computing it using the Laplace transform. Assume $u(t) \equiv 0$, i.e., that $u(t) = 0$ for all values of t. In this case, by (2) we see that the state trajectory is given by

$$x(t) = e^{At}x(0).$$

On the other hand, considering (1) and taking the Laplace transform of the differential equation gives

$$sX(s) - x(0) = AX(s).$$

Solving this system for $X(s)$, and taking the inverse Laplace transform gives that

$$x(t) = \mathcal{L}^{-1}[X(s)] = \mathcal{L}^{-1}[(sI - A)^{-1}x(0)] = \mathcal{L}^{-1}[(sI - A)^{-1}]x(0).$$

By comparing the two expression, and by the uniqueness of both the solution to the ode and the matrix exponential, we get that

$$e^{At} = \mathcal{L}^{-1}[(sI - A)^{-1}].$$

We did an exercise on this during the exercise session. Partial fractional expansion was used in order to get the expressions “on standard form”, which can then be found in a table over the Laplace transform in order to get the expression for the matrix exponential.

iii) computing it using diagonalization or Jordan form. This we did not have time for during the first exercise session, and I will therefore summarize the method here.

In short: any matrix can be written in Jordan form. That means that it can be written as

$$A = TJT^{-1} \tag{3}$$

where J has the form

$$J = \begin{bmatrix} J_1 & 0 & \dots & 0 \\ 0 & J_2 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & J_k \end{bmatrix},$$

and where the J_is have are matrices of the form

$$J_i = \begin{bmatrix} \lambda_i & 1 & 0 & \dots & 0 \\ 0 & \lambda_i & 1 & \dots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & & 0 & \lambda_i \end{bmatrix},$$

where λ_i is an eigenvalue of A. Note that the same eigenvalue can occur in different submatrices J_i and J_ℓ. Also note that the diagonalization of a matrix is a special kind of Jordan form where each submatrix J_i is of size 1×1 (and thus only contain an eigenvalue). Now, since J is block-diagonal, by putting (3) into the definition of the matrix exponential, we get

$$e^{At} = \sum_{k=0}^{\infty} \frac{t^k}{k!}(TJT^{-1})^k = T \left(\sum_{k=0}^{\infty} \frac{t^k}{k!} J^k \right) T^{-1} = T \operatorname{diag}(e^{J_1 t}, \dots, e^{J_k t})\, T^{-1},$$

and due to the special structure of the matrices J_i the corresponding matrix exponentials $e^{J_i t}$ can be computed. In order to see how this is done in practise we will do an example.

Exercise 1.5

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -4 & 4 \\ 0 & -1 & 0 \end{bmatrix}$$

We start by finding the eigenvalues of A.

$$0 = p_A(\lambda) = \det(\lambda I - A) = \begin{vmatrix} \lambda+1 & 0 & 0 \\ 0 & \lambda+4 & -4 \\ 0 & 1 & \lambda \end{vmatrix} = (\lambda+1)\Big((\lambda+4)\lambda+4\Big) = (\lambda+1)(\lambda+2)^2.$$

This gives that $\lambda_1 = -1$ and $\lambda_2 = \lambda_3 = -2$.

An eigenvector to λ_1 is given by

$$(\lambda_1 I - A)v_1 = 0 \quad \Longleftrightarrow \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & -4 \\ 0 & 1 & -1 \end{bmatrix} v_1 = 0,$$

and we see that $v_1 = [1\ 0\ 0]^T$ is an eigenvector.

Eigenvalues for λ_2 and λ_3 are sought in a similar manner:

$$(\lambda_2 I - A) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & -4 \\ 0 & 1 & -2 \end{bmatrix}.$$

However, from this one can see that $\dim(\ker(A)) = 1$, and so the geometric multiplicity of the eigenvalues is 1, while the algebraic multiplicity is 2 (it is a double root; $\lambda_2 = \lambda_3$). Hence we need to consider generalized eigenvalues:

$$(\lambda_2 I - A)v_2 = 0 \tag{4}$$
$$(\lambda_2 I - A)v_3 = v_2. \tag{5}$$

In general, if the algebraic multiplicity of λ is m and the geometric multiplicity is 1, one consider

$$\begin{aligned} (\lambda I - A)v_{\ell_1} &= 0 \\ (\lambda I - A)v_{\ell_2} &= v_{\ell_1} \\ &\vdots \\ (\lambda I - A)v_{\ell_m} &= v_{\ell_{m-1}}. \end{aligned}$$

From (4) we get

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & -4 \\ 0 & 1 & -2 \end{bmatrix} v_2 = 0 \Longrightarrow v_2 = \alpha \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix},$$

and putting this into (5) gives

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & -4 \\ 0 & 1 & -2 \end{bmatrix} v_3 = \alpha \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}.$$

One solution to this is $\alpha = -1$ and $v_3 = [0\ 1\ 0]^T$, which gives $v_2 = [0\ -2\ -1]^T$.

Hence, the Jordan form of A is

$$A = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \operatorname{diag}(J_1, J_2) \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & -2 \end{bmatrix},$$

where the last step is simply putting in the values for v_1, v_2, v_3, $J_1 = [1]$ and $J_2 = \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix}$, and inverting a 3×3 matrix.

Now, using all of this we get

$$e^{At} = Te^{Jt}T^{-1} = T \operatorname{diag}(e^{J_1 t}, e^{J_2 t})\, T^{-1}$$

where

$$e^{J_1 t} = e^{-t}$$

and

$$e^{J_2 t} = \underbrace{e^{-2It+St} = e^{-2It}e^{St}}_{\text{Motivate for yourself why this holds}}, \quad \text{where } S = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Now, from the second exercise we did on the first session we know that

$$e^{-2It} = \begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{-2t} \end{bmatrix}$$

and from the first one we know that

$$e^{St} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix},$$

which gives

$$e^{J_2 t} = \begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} e^{-2t} & te^{-2t} \\ 0 & e^{-2t} \end{bmatrix}.$$

Finally,

$$e^{At} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & e^{-2t} & te^{-2t} \\ 0 & 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & -2 \end{bmatrix} = \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & e^{-2t} - 2te^{-2t} & 4te^{-2t} \\ 0 & -te^{-2t} & 2te^{-2t} + e^{-2t} \end{bmatrix}.$$

Final remark: this exercise was solve in this way for educational purposes. However, it could have been solved in a more clever way by first noting that one is in fact faced with two decoupled systems:

$$\begin{aligned} \dot{x}_1 &= -x_1 \\ \begin{bmatrix} \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} &= \begin{bmatrix} -4 & 4 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ x_3 \end{bmatrix}. \end{aligned}$$