# SYSTEMTEORI - ÖVNING 1

GIANANTONIO BORTOLIN AND RYOZO NAGAMUNE

In this exercise, we will learn how to solve the following linear differential equation:

$$\dot{x}(t) = A(t)x(t), \quad x(t_0) = x_0, \quad x(t) \in \mathbb{R}^n, \quad A(t) \in \mathbb{R}^{n\times n} \tag{0.1}$$

The equation describes an autonomous linear dynamical system. We will consider the following two cases:

(1) Time invariant case, i.e., $A(t) \equiv A$ (constant matrix).
(2) Time varying case, i.e., $A(t)$ depends on the time $t$.

## 1. Solution of linear time invariant autonomous systems

Let us consider to solve

$$\dot{x}(t) = Ax(t), \quad x(t_0) = x_0, \quad x(t) \in \mathbb{R}^n, \quad A \in \mathbb{R}^{n\times n} \quad (A \text{ constant}) \tag{1.1}$$

The solution to this equation can be explicitly written as

$$x(t) = e^{A(t-t_0)}x_0. \tag{1.2}$$

Now, the question is how to compute the matrix exponential $e^{At}$. Here, we will present four methods, using

- definition of the matrix exponential,
- diagonalization or Jordan form,
- Laplace transform,
- Cayley-Hamilton Theorem.

*Remark* 1. In Matlab, we can compute the matrix exponential $e^A$ numerically with the command `expm`.

### 1.1. Using the definition of the matrix exponential.

Let $A$ be a $n\times n$ matrix. Then, given any $t \in \mathbb{R}$, the exponential of a matrix is defined in the following way:

$$e^{At} := I + \frac{t}{1!}A + \frac{t^2}{2!}A^2 + \cdots = \sum_{k=0}^{\infty} \frac{(At)^k}{k!}.$$

For some simple matrices $A$, it is easy to compute the matrix exponential by using this definition.

**Example 1.** *(Nilpotent case)* Suppose that

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

It is easy to see that $A^k = 0$ for $k \geq 2$. Therefore,

$$e^{At} = I + At = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

**Example 2.** *(Diagonal case)* Suppose that

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}.$$

It is easy to see that

$$e^{At} = \sum_{k=0}^{\infty} \frac{(At)^k}{k!} = \begin{pmatrix} \sum_{k=0}^{\infty} \frac{(2t)^k}{k!} & 0 \\ 0 & \sum_{k=0}^{\infty} \frac{(3t)^k}{k!} \end{pmatrix} = \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{3t} \end{pmatrix}.$$

In general, if the matrix $A$ is **diagonal**, that is $A = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$, then it is easy to prove that:

$$e^{At} = \mathrm{diag}(e^{\lambda_1 t}, ..., e^{\lambda_n t}). \tag{1.3}$$

More generally, if the matrix $A$ is **block-diagonal**, that is,

$$A = \begin{pmatrix} A_1 & 0 & \dots & 0 \\ 0 & A_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_m \end{pmatrix} = \mathrm{diag}(A_1, ..., A_m)$$

then,

$$A^k = \mathrm{diag}(A_1^k, ..., A_m^k) \quad \text{and} \quad e^{At} = \mathrm{diag}(e^{A_1 t}, ..., e^{A_m t}).$$

1.2. **Using diagonalization or Jordan form.** It is well-known that any matrix $A$ can always be transformed into a block diagonal form $J$ by a similarity transformation with an appropriate nonsingular matrix $T$ as

$$J = T^{-1}AT, \text{ or } A = TJT^{-1}. \tag{1.4}$$

1.2.1. *Diagonalizable $A$.* Let us suppose that the matrix $A$ has $n$ independent eigenvectors, $v_i$. Then, there exists a matrix $T = [v_1 ... v_n]$ such that:

$$T^{-1}AT = D = \mathrm{diag}(\lambda_1, ..., \lambda_n), \tag{1.5}$$

where $\lambda_i$ are the eigenvalues of $A$. In this case, it is easy to show that:

$$e^{At} = e^{TDT^{-1}t} = T\mathrm{diag}(e^{\lambda_1 t}, ..., e^{\lambda_n t})T^{-1}. \tag{1.6}$$

**Example 3.** Let

$$A = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}$$

To compute $e^{At}$, note that the characteristic polynomial of $A$ is:

$$\chi_A(\lambda) = \det(\lambda I - A) = (\lambda + 1)(\lambda + 2) \Rightarrow \lambda_1 = -1, \lambda_2 = -2$$

So, the eigenvalues of $A$ are $-1$ and $-2$, and the corresponding eigenvectors are $[1, -1]^T$ and $[1, -2]^T$, respectively. Thus, $A$ can be diagonalized by a similarity transformation as

$$A = \underbrace{\begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix}}_{T} \underbrace{\begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}}_{D} \underbrace{\begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix}^{-1}}_{T^{-1}}. \tag{1.7}$$

Using (1.6),

$$e^{At} = \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix}^{-1} \tag{1.8}$$

$$= \begin{pmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{pmatrix}. \tag{1.9}$$

1.2.2. *Jordan form for A.* It is always possible to find a nonsingular matrix $T$ such that $J = T^{-1}AT$ is in Jordan canonical form:

$$J = \begin{pmatrix} \lambda_1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & \ddots & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & \lambda_1 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \\ 0 & 0 & \dots & 0 & \lambda_k & 1 & \dots \\ 0 & 0 & \dots & 0 & 0 & \ddots & 1 \\ 0 & 0 & \dots & 0 & 0 & 0 & \lambda_k \end{pmatrix} = \mathrm{diag}(J_1, ..., J_k)$$

and so the problem is reduced to calculate the exponential of each Jordan block $J_j$. The Jordan blocks $J_j$ have the following form:

$$J_j = \begin{pmatrix} \lambda_j & 1 & 0 & 0 & \dots \\ 0 & \lambda_j & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \\ 0 & 0 & \dots & \lambda_j & 1 \\ 0 & 0 & 0 & \dots & \lambda_j \end{pmatrix} = \lambda_j I + S_j$$

where $S_j$ is a shift matrix:

$$S_j = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \\ 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

with the property that $S_j^i = 0$ for $i \geq d_j$ where $d_j$ is the dimension of $S_j$. Therefore, we have that:

$$e^{J_j t} = e^{\lambda_j t} e^{S_j t} = e^{\lambda_j t} \left( I + S_j t + \frac{(S_j t)^2}{2} + \ldots + \frac{(S_j t)^{d_j - 1}}{(d_j - 1)!} \right) \tag{1.10}$$

$$= e^{\lambda_j t} \begin{pmatrix} 1 & t & t^2/2 & \cdots & \frac{t^{d_j-1}}{(d_j-1)!} \\ 0 & 1 & t & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & t^2/2 \\ \vdots & \ddots & \ddots & \ddots & t \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix}.$$

*Remark* 2. To learn how to obtain a Jordan form from a general matrix, see “Jordandekomposition” in “Kopior på overheadbilder” by C. Trygger.

**Example 4.** Let $A$ be

$$A = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix} \tag{1.11}$$

which is already in a Jordan form. From (1.10), we have

$$e^{At} = e^{-t} \begin{pmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} \tag{1.12}$$

1.3. **Using Laplace transform.** The matrix exponential $e^{At}$ can be obtained by

$$e^{At} = \mathcal{L}^{-1}\{(sI - A)^{-1}\}, \tag{1.13}$$

where $\mathcal{L}^{-1}$ means the inverse Laplace transform.

*Remark* 3. See the table for the Laplace transform in, for example, "BETA: Mathematics Handbook for Science and Engineering."

**Example 5.** Let us again consider the matrix

$$A = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}. \tag{1.14}$$

Then,

$$(sI - A)^{-1} = \begin{pmatrix} s & -1 \\ 2 & s+3 \end{pmatrix}^{-1} = \frac{1}{s^2 + 3s + 2} \begin{pmatrix} s+3 & 1 \\ -2 & s \end{pmatrix}. \tag{1.15}$$

Now, we decompose this with a *partial fraction expansion*:

$$\frac{1}{s^2 + 3s + 2} \begin{pmatrix} s+3 & 1 \\ -2 & s \end{pmatrix} = \frac{1}{s+1} \begin{pmatrix} 2 & 1 \\ -2 & -1 \end{pmatrix} + \frac{1}{s+2} \begin{pmatrix} -1 & -1 \\ 2 & 2 \end{pmatrix}. \tag{1.16}$$

Therefore,

$$\begin{aligned} e^{At} &= \mathcal{L}^{-1}\left\{(sI - A)^{-1}\right\} && (1.17) \\ &= \mathcal{L}^{-1}\left(\frac{1}{s+1}\right) \begin{pmatrix} 2 & 1 \\ -2 & -1 \end{pmatrix} + \mathcal{L}^{-1}\left(\frac{1}{s+2}\right) \begin{pmatrix} -1 & -1 \\ 2 & 2 \end{pmatrix} && (1.18) \\ &= e^{-t} \begin{pmatrix} 2 & 1 \\ -2 & -1 \end{pmatrix} + e^{-2t} \begin{pmatrix} -1 & -1 \\ 2 & 2 \end{pmatrix}. && (1.19) \end{aligned}$$

**Example 6.** Consider the matrix

$$A = \begin{pmatrix} \sigma & \omega \\ -\omega & \sigma \end{pmatrix}. \tag{1.20}$$

Then,

$$(sI - A)^{-1} = \begin{pmatrix} s-\sigma & -\omega \\ \omega & s-\sigma \end{pmatrix}^{-1} = \frac{1}{(s-\sigma)^2 + \omega^2} \begin{pmatrix} s-\sigma & \omega \\ -\omega & s-\sigma \end{pmatrix}. \tag{1.21}$$

Using the inverse Laplace transform formula, we have

$$e^{At} = \mathcal{L}^{-1}\left\{(sI - A)^{-1}\right\} = \begin{pmatrix} e^{\sigma t}\cos(\omega t) & e^{\sigma t}\sin(\omega t) \\ -e^{\sigma t}\sin(\omega t) & e^{\sigma t}\cos(\omega t) \end{pmatrix}. \tag{1.22}$$

1.4. **Using Cayley-Hamilton Theorem.**

**Theorem 1.** *Any square matrix $A$ satisfies its characteristic polynomial.*

In other words, given the characteristic polynomial of the matrix $A \in \mathbb{R}^{n\times n}$:

$$\chi_A(\lambda) = \det(\lambda I - A) = \lambda^n + a_1 + \lambda^{n-1} + \ldots + a_n$$

then:

$$\chi_A(A) = A^n + a_1 A^{n-1} + \ldots + a_n I = 0$$

*Remark* 4. The theorem has two important consequences:

**1:** $A^n = -a_1 A^{n-1} - a_2 A^{n-2} - \ldots - a_n I$.

**2:** Every matrix polynomial $\psi(A)$ of order $n+i, i \geq 0$ can be expressed by an $(n-1)$-order polynomial. Another implication is that $e^{At}$ is an infinity order polynomial which can also be expressed as an $(n-1)$-order polynomial.

**Theorem 2.** *Let $\lambda_i, i = 1,..,m$ be the eigenvalues of an $n\times n$ matrix of multiplicity $n_i$, i.e.*

$$\chi_A(\lambda) = \prod_{i=1}^{m} (\lambda - \lambda_i)^{n_i} \quad \textit{and} \quad \sum_{i=1}^{m} n_i = n$$

*Let also $f(\lambda)$ and $g(\lambda)$ be polynomials of $\lambda$ such that:*

$$\frac{d^k}{d\lambda^k} f(\lambda)|_{\lambda=\lambda_i} = \frac{d^k}{d\lambda^k} g(\lambda)|_{\lambda=\lambda_i}, \quad \forall \quad i = 1, ..., m, \quad k = 0, ..., n_i - 1 \tag{1.23}$$

*Then $f(A) = g(A)$.*

*Remark* 5. The previous theorem can be used to find a $(n-1)$-order polynomial $g(\lambda)$ corresponding to a (possibly high order) polynomial function $f(A)$. To this end, note that Eq. (1.23) constitutes $n$ equations, from which $n$ coefficients of $g(\lambda)$ can be found.

**Example 7.** In this simple example we are going to see how the Cayley-Hamilton theorem is used to compute a matrix exponential. Consider again the matrix

$$A = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix},$$

where $n = 2$. To compute $e^{At}$, note that the characteristic polynomial of $A$ is:

$$\chi_A(\lambda) = \det(\lambda I - A) = (\lambda + 1)(\lambda + 2) \Rightarrow \lambda_1 = -1, \lambda_2 = -2$$

Let $f(\lambda) := e^{\lambda} t$. We are looking for a first-order polynomial $g(\lambda) = g_1\lambda + g_0$ such that Eq. (1.23) is satisfied. That is:

$$g(\lambda_i) = e^{\lambda_i}, \quad i = 1, 2$$

The coefficients $g_0$ and $g_1$ can be found from:

$$\begin{cases} g_1\lambda_1 + g_0 &= e^{\lambda_1 t} \\ g_1\lambda_2 + g_0 &= e^{\lambda_2 t} \end{cases} \Leftrightarrow \begin{cases} -g_1 + g_0 &= e^{-t} \\ -2g_1 + g_0 &= e^{-2t} \end{cases} \Leftrightarrow \begin{cases} g_0 &= 2e^{-t} - e^{-2t} \\ g_1 &= e^{-t} - e^{-2t} \end{cases}$$

Due to Theorem 2, we have $f(A) = g(A)$, that is,

$$e^{At} = g_1 A + g_0 I = \begin{pmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2(e^{-t} - e^{-2t}) & -(e^{-t} - 2e^{-2t}) \end{pmatrix}.$$

**Example 8.** Let us consider the following matrix $A$:

$$A = \begin{pmatrix} 0 & 1 & & & \dots & & \\ 0 & 0 & & & \dots & & \\ & \dots & 0 & 1 & \dots & & \\ & \dots & -1 & 0 & \dots & & \\ & \dots & & & 0 & 0 & -2 \\ & \dots & & & 0 & 1 & 0 \\ & \dots & & & 1 & 0 & 3 \end{pmatrix}$$

We want to determine the transition matrix $\Phi(t,s)$.

First of all, we should note that $A$ is a constant block diagonal matrix, that is $A = \mathrm{diag}(A_1, A_2, A_3)$ where:

$$A_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & 0 & -2 \\ 0 & 1 & 0 \\ 1 & 0 & 3 \end{pmatrix}$$

and so we have that $\Phi(t,s) = e^{A(t-s)} = \mathrm{diag}(e^{A_1(t-s)}, e^{A_2(t-s)}, e^{A_3(t-s)})$. Hence, let us compute each block one by one:

**1:** $A_1$ is a shift matrix, that is $A_1^2 = 0$. Therefore:

$$e^{A_1 t} = I + A_1 t$$

**2:** $A_2$ is a typical case of oscillatory solution. In general, given a $2 \times 2$ matrix $F$ which has a couple of complex conjugate eigenvalues, $\lambda_{1,2} = \sigma \pm j\omega$, it is always possible to find a change of basis so that $F$ can be written in the form:

$$F = \begin{pmatrix} \sigma & \omega \\ -\omega & \sigma \end{pmatrix}$$

From Example 6,

$$e^{Ft} = \begin{pmatrix} e^{\sigma t}\cos(\omega t) & e^{\sigma t}\sin(\omega t) \\ -e^{\sigma t}\sin(\omega t) & e^{\sigma t}\cos(\omega t) \end{pmatrix}$$

Therefore:

$$e^{A_2 t} = \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix}$$

**3:** To compute $e^{A_3 t}$ we can use the Cayley-Hamilton theorem. First, let us compute the eigenvalues of $A_3$:

$$\det(\lambda I - A) = (\lambda - 1)^2(\lambda - 2)$$

and so we have the eigenvalue $\lambda_1 = 1$ with multiplicity 2, and the eigenvalue $\lambda_2 = 2$ with multiplicity 1.

We are thus looking for a 2-nd order polynomial $g(\lambda) = g_2\lambda^2 + g_1\lambda + g_0$ such that:

$$g(\lambda_1) = e^{\lambda_1 t}, \qquad \frac{d}{d\lambda} g(\lambda)|_{\lambda=\lambda_1} = \frac{d}{d\lambda} e^{\lambda t}|_{\lambda=\lambda_1}, \quad \text{and} \quad g(\lambda_2) = e^{\lambda_2 t}$$

By substituting the numerical values we get:

$$\begin{cases} g_2 + g_1 + g_0 & = & e^t \\ 2g_2 + g_1 & = & te^t \\ 4g_2 + 2g_1 + g_0 & = & e^{2t} \end{cases} \quad \rightarrow \begin{cases} g_0 & = & -2te^t + e^{2t} \\ g_1 & = & 2e^t + 3te^t - 2e^{2t} \\ g_2 & = & -e^t - te^t + e^{2t} \end{cases}$$

Finally, we have that:

$$e^{A_3 t} = (-2te^t + e^{2t})I + (3te^t + 2e^2 - 2e^{2t})A + (e^{2t} - e^t - te^t)A^2$$

## 2. Solution of linear time varying autonomous systems

In the time varying case, the solution to the linear system with input

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \tag{2.1}$$

is written as

$$x(t) = \Phi(t, t_0)x(0) + \int_{t_0}^{t} \Phi(t, \tau)B(\tau)u(\tau)d\tau, \tag{2.2}$$

where $\Phi(t, s)$ is a transition matrix. The question is how to compute this transition matrix.

If matrices $A(t)$ and $\int_s^t A(\tau)d\tau$ commute, then $\Phi(t, s)$ can be computed by

$$\Phi(t, s) = e^{\int_s^t A(\tau)d\tau}. \tag{2.3}$$

*Remark* 6. Note that, in the time invariant case (constant $A$), this reduces to

$$\Phi(t, s) = e^{A(t-s)}. \tag{2.4}$$

**Example 9.** (Exercise (1.13)) Let us consider the following linear time varying system (LTV):

$$\dot{x}(t) = \begin{pmatrix} \cos(t) - 4/t & -1/t \\ 4/t & \cos(t) \end{pmatrix} x(t) = A(t)x(t)$$

We want to determine the transition matrix $\Phi(t, s)$.

We can express $A(t)$ as:

$$A(t) = \cos(t) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{t} \begin{pmatrix} -4 & -1 \\ 4 & 0 \end{pmatrix} = I\cos(t) + K\frac{1}{t}$$

First, let us compute the integral of $A(t)$:

$$B(t, s) = \int_s^t (I\cos(\tau) + \frac{1}{\tau}K)d\tau = (\sin(t) - \sin(s))I + \ln(\frac{t}{s})K$$

Note that $B(t, s)$ and $A$ commute, that is $A(t)B(t, s) = B(t, s)A(t)$. In such a case, see remark 2.1.3 in the compendium, the transition matrix is:

$$\Phi(t, s) = \exp(B(t, s)) = \exp((\sin(t) - \sin(s))I + \ln(\frac{t}{s})K)$$

Note that $K$ and $I$ commute. To calculate $\exp(\ln(\frac{t}{s})K)$ we can use the Laplace transform:

$$e^{xK} = \mathcal{L}^{-1}\left\{ \begin{pmatrix} s+4 & 1 \\ -4 & s \end{pmatrix}^{-1} \right\} = \begin{pmatrix} (1-2x) & -x \\ 4x & 1+2x \end{pmatrix} e^{-2x}$$

So, by setting $x = \ln(t/s)$ we get:

$$\Phi(t, s) = \exp(\sin(t) - \sin(s))\frac{s^2}{t^2} \begin{pmatrix} 1 - 2\ln(t/s) & -\ln(t/s) \\ 4\ln(t/s) & 1 + 2\ln(t/s) \end{pmatrix}$$

The following is an example where matrices $A(t)$ and $\int_s^t A(\tau)d\tau$ do *not* commute.

**Example 10.** (Exercise (1.14)) Let us consider the following LTV system:

$$\dot{x}(t) = \begin{pmatrix} 0 & 0 \\ t & 1/t \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t) = A(t)x(t) + Bu(t)$$

We want to determine the transition matrix $\Phi(t,s)$.

Like before, let us compute the following integral:

$$B(t,s) = \int_s^t A(\tau)d\tau = \begin{pmatrix} 0 & 0 \\ (t^2-s^2)/2 & \ln(\frac{t}{s}) \end{pmatrix}$$

Do $B(t,s)$ and $A$ commute?

$$\begin{aligned} [A(t), B(t,s)] &:= A(t)B(t,s) - B(t,s)A(t) \\ &= \begin{pmatrix} 0 & 0 \\ \frac{(t^2-s^2)}{(2t)} & \frac{1}{t}\ln(\frac{t}{s}) \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ t\ln(\frac{t}{s}) & \frac{1}{t}\ln(\frac{t}{s}) \end{pmatrix} \neq 0 \end{aligned}$$

Unfortunately in this case they do not commute, and so we cannot use the formula (2.3) to compute the transition matrix. However, in this example, we can compute the transition matrix in the following way.

$$\dot{x}_1 = 0 \rightarrow x_1 = x_{10}$$
$$\dot{x}_2 = tx_1 + \frac{1}{t}x_2$$

The solution of $x_2(t)$ can be computed using standard solution formula (2.2), with the transition matrix for the system equation for $x_2$:

$$\Phi_2(t,s) := e^{\int_s^t 1/\tau d\tau} = t/s. \tag{2.5}$$

Therefore,

$$\begin{aligned} x_2(t) &= \Phi_2(t,t_0)x_{20} + \int_{t_0}^t \Phi_2(t,\tau)\cdot\tau x_{10}d\tau, \quad (2.6) \\ &= \frac{t}{t_0}x_{20} + t(t-t_0)x_{10}. \quad (2.7) \end{aligned}$$

We have

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ t(t-t_0) & t/t_0 \end{pmatrix} \begin{pmatrix} x_{10} \\ x_{20} \end{pmatrix}. \tag{2.8}$$

Therefore, the transition matrix is given by:

$$\Phi(t,s) = \begin{pmatrix} 1 & 0 \\ t(t-s) & t/s \end{pmatrix}.$$