SYSTEMTEORI - ÖVNING 4: REALIZATION THEORY

1. Short review of Realization Theory

We have seen that a linear system described by system matrices (A, B, C, D) defines an input-output mapping through the transfer matrix $R(s)$:

$$Y(s) = R(s)U(s), \quad R(s) := C(sI - A)^{-1}B + D, \tag{1.1}$$

where $U(s)$ and $Y(s)$ are the Laplace transforms of the input and output vectors. In section 5.1 of the compendium, it is proved that the transfer function $R(s)$ is a matrix of proper rational functions. It is strictly proper if $D = 0$.

In general, the transfer function is a useful and compact way of describing a linear system. Besides, there are quite standard and efficient ways of estimating transfer functions from input-output data.

However, in many situations, it is more convenient to have state space representations, instead of transfer function representations. Therefore, the problem we want to attack is the following.

Realization problem: Given a transfer matrix $R(s)$, determine a state space representation (A, B, C, D) such that:

$$R(s) = C(sI - A)^{-1}B + D$$

Every state space representation (A, B, C, D) that satisfies such problem is called a realization of $R(s)$.

It can be proved that there are infinitely many matrices (A, B, C, D) that are realizations of the same transfer function $R(s)$. Two important ones are the standard reachable realization, and the standard observable realization, see Theorem 5.1.5 in the compendium. Suppose $R(s)$ is a $m \times k$ strictly proper transfer matrix, and let $\chi(s)$ be the least common denominator of the elements of $R(s)$:

$$\chi(s) = s^r + a_1 s^{r-1} + \cdots + a_r. \tag{1.2}$$

Define the $m \times k$ matrices N_i as:

$$\chi(s)R(s) = N_0 + N_1 s + N_2 s^2 + \cdots + N_{r-1}s^{r-1}. \tag{1.3}$$

Besides, define the Markov parameters, R_i, as the coefficients of the Laurent-series of $R(s)$:

$$R(s) = R_1 s^{-1} + R_2 s^{-2} + R_3 s^{-3} + \cdots . \tag{1.4}$$

Then, we have that:

- **The standard reachable realization**:

$$A = \begin{bmatrix} 0 & I_k & 0 & \cdots & 0 \\ 0 & 0 & I_k & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & I_k \\ -a_r I_k & -a_{r-1} I_k & -a_{r-2} I_k & \cdots & -a_1 I_k \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ I_k \end{bmatrix},$$

$$C = \begin{bmatrix} N_0 & N_1 & \cdots & N_{r-1} \end{bmatrix}.$$

- **The standard observable realization**:

$$A = \begin{bmatrix} 0 & I_m & 0 & \cdots & 0 \\ 0 & 0 & I_m & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & I_m \\ -a_r I_m & -a_{r-1} I_m & -a_{r-2} I_m & \cdots & -a_1 I_m \end{bmatrix}, \quad B = \begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_r \end{bmatrix},$$

$$C = \begin{bmatrix} I_m & 0 & \cdots & 0 \end{bmatrix}.$$

In general, it is of particular interest determining minimal realizations, that is state space representations of minimal dimension. There are different reasons why the minimal realization problem deserves to be studied:

- In order to analyze a system, it is advantageous to have a compact description of it.
- Since the minimal realization is both observable and reachable (see Theorem 5.2.6 in the compendium), it is a good basis for designing an observer and a controller.
- The minimal realization problem can be solved very elegantly using linear algebra methods.

The methods to determine a minimal realization can be classified in two groups:

- The first method starts from a non-minimal realization, and then reduce it, by using for example Kalman decomposition, see Theorem 5.2.1 in the compendium.
- The second method starts from the Markov parameters, and obtain the minimal realization by suitable transformation of the resulting Hankel matrix. An example is the Ho-Kalman algorithm described in Theorem 5.3.2 of the compendium, and briefly described next.

Ho-Kalman algorithm:

Step 1: Using Markov parameters R_i of $R(s)$, construct a Hankel matrix H_r:

$$H_r = \begin{bmatrix} R_1 & R_2 & \cdots & R_r \\ R_2 & R_3 & \cdots & R_{r+1} \\ \vdots & \vdots & \ddots & \vdots \\ R_r & R_{r+1} & \cdots & R_{2r-1} \end{bmatrix}, \tag{1.5}$$

where r is the degree of the least common denominator of $R(s)$.

Step 2: Suppose H_r has rank n. Find two nonsingular matrices P and Q satisfying

$$P H_r Q = \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix}.$$

(Numerically, this step can be done by the singular value decomposition.)

Step 3: Let $\sigma(\cdot)$ be the shift operator. Obtain (A, B, C) by

$$\begin{aligned}
A &= [I_n|0]P\sigma(H_r)Q\begin{bmatrix} I_n \\ \hline 0 \end{bmatrix}, \\
B &= [I_n|0]PH_r\begin{bmatrix} I_k \\ 0 \\ \vdots \\ 0 \end{bmatrix} = [I_n|0]P\begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_r \end{bmatrix}, \\
C &= [I_m, 0, \cdots, 0]H_rQ\begin{bmatrix} I_n \\ \hline 0 \end{bmatrix} = [R_1, R_2, \cdots, R_r]Q\begin{bmatrix} I_n \\ \hline 0 \end{bmatrix}.
\end{aligned}$$

Then, (A, B, C) is a minimal realization.

2. Examples

Exercise 4.1. Consider the following transfer function:

$$g(s) = \frac{2s-4}{s^3 - 7s + 6}$$

(a): Determine the standard reachable realization.
(b): Determine the standard observable realization.
(c): Determine a minimal realization.

Note that the least common denominator of $g(s)$ is:

$$\chi(s) = s^3 - 7s + 6,$$

and thus, $r = 3$, $a_1 = 0$, $a_2 = -7$, $a_3 = 6$. Consequently,

$$g(s)\chi(s) = 2s - 4 \Rightarrow N_0 = -4,\ N_1 = 2,\ N_2 = 0.$$

Note also that $D = 0$, since $g(s)$ is strictly proper.

(a): From the strictly proper transfer function $g(s)$, the standard reachable realization can be immediately obtained as

$$\begin{aligned}
A &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & 7 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \\
C &= \begin{bmatrix} -4 & 2 & 0 \end{bmatrix}, \quad D = 0.
\end{aligned}$$

This realization is reachable by construction. We can verify it by checking the reachability matrix Γ:

$$\Gamma = [B, AB, A^2B] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 7 \end{bmatrix},$$

which is clearly of full rank. Is this realization also observable?

$$\Omega = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} -4 & 2 & 0 \\ 0 & -4 & 2 \\ -12 & 14 & -4 \end{bmatrix} \sim \begin{bmatrix} -4 & 2 & 0 \\ 0 & -4 & 2 \\ 0 & 8 & -4 \end{bmatrix} \sim \begin{bmatrix} -4 & 2 & 0 \\ 0 & -4 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

which is not full rank. So, this realization is not minimal.

(b): The Markov parameters can be determined in the following way.

$$g(s) = \frac{2s-4}{s^3-7s+6} = \frac{2(s-2)}{(s-2)(s-1)(s+3)} = \frac{2}{(s-1)(s+3)}$$

$$\begin{aligned}
\frac{1}{s-1} &= \frac{1}{s}\frac{1}{1-1/s} = \frac{1}{s}(1+s^{-1}+s^{-2}+\cdots) = s^{-1}+s^{-2}+\cdots \\
\frac{1}{s+3} &= \frac{1}{s}\frac{1}{1+3/s} = \frac{1}{s}(1-3s^{-1}+9s^{-2}-\cdots) = s^{-1}-3s^{-2}+9s^{-3}-\cdots
\end{aligned}$$

$g(s) = 2(s^{-1}+s^{-2}+\cdots)(s^{-1}-3s^{-2}+9s^{-3}-\cdots) = 2s^{-2}-4s^{-3}+\cdots$
$\Rightarrow R_1 = 0,\ R_2 = 2,\ R_3 = -4.$

Using Markov parameters, the standard observable realization is given by:

$$\begin{aligned}
A &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & 7 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 2 \\ -4 \end{bmatrix}, \\
C &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \qquad D = 0.
\end{aligned}$$

This realization is observable by construction. Is it also reachable? No, because from part a) we got that the McMillan degree is less than 3.

(c): Let us simplify the transfer function:

$$g(s) = \frac{2}{(s-1)(s+3)}$$

and let us determine the reachable canonical form:

$$\chi(s) = (s-1)(s+3) = s^2+2s-3 \Rightarrow a_1 = 2, a_2 = -3$$

and

$$g(s)\chi(s) = 2 \quad \Rightarrow N_0 = 2, N_1 = 0$$

So, the reachable canonical form is:

$$\begin{aligned}
A &= \begin{bmatrix} 0 & 1 \\ 3 & -2 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
C &= \begin{bmatrix} 2 & 0 \end{bmatrix} \quad D = 0
\end{aligned}$$

Is this realization minimal? It is reachable by construction, and so we have only to check if it is observable:

$$\Omega = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

which is clearly full rank. Therefore, this realization is minimal.

We will see that, *for SISO systems, a realization is minimal if and only if there are no pole-zero cancellation in the corresponding transfer function.*

Theorem: *Consider a SISO system (A, B, C). The system is completely reachable and observable iff there are no common roots between det$(sI-A)$ and Cadj$(sI-A)B$.*

Proof. The transfer function associated to the system (A, B, C) is given by:

$$R(s) = C(sI-A)^{-1}B = \frac{C\text{adj}(sI-A)B}{\det(sI-A)}$$

Let us assume that there are no common roots between the two polynomials, and that the system is not minimal, that is not reachable and/or not observable. But then, there exists a system defined by $(\tilde{A}, \tilde{B}, \tilde{C})$ which has a smaller dimension, but the same transfer function: The transfer function associated to the system (A, B, C) is given by:

$$R(s) = \frac{C\text{adj}(sI - A)B}{\det(sI - A)} = \frac{\tilde{C}\text{adj}(sI - \tilde{A})\tilde{B}}{\det(sI - \tilde{A})}.$$

But this implies that there are common roots between $\det(sI - A)$ and $C\text{adj}(sI - A)B$, which contradicts our assumption.

Now, let us suppose that (A, B, C) is minimal, but there exist common roots between the two polynomials. In such case the transfer function:

$$R(s) = C(sI - A)^{-1}B = \frac{C\text{adj}(sI - A)B}{\det(sI - A)}$$

could be written as the ratio between two polynomials of smaller degrees:

$$R(s) = \frac{p(s)}{d(s)}$$

with degree of $d(s) <$ degree $\det(sI - A)$. But then we could find a realization of $R(s)$ of the same degree of $d(s)$, and so of smaller dimension of the original one. But this contradicts our assumption that (A, B, C) is minimal. Q.E.D.

Remark: In the scalar case, the standard reachable realization and the standard observable realization are also minimal if all the common roots between the numerator and the denominator of the transfer function $R(s)$ have been removed before realization. Unfortunately, in the MIMO case, the situation is more complicated and in order to obtain a minimal realization, the Kalman decomposition or Ho's algorithm have to be used.

Exercise 4.6 (modified). Consider the transfer matrix

$$R(s) = \begin{bmatrix} \frac{1}{s+1} & \frac{s+3}{s+1} \\ \frac{-1}{(s+1)(s+2)} & \frac{1}{s+2} \end{bmatrix}$$

- **a):** Determine the standard reachable form
- **b):** Is the standard reachable form minimal?
- **c):** Determine the standard observable form
- **d):** Is the standard observable form minimal?
- **e):** Determine the McMillan degree
- **f):** Determine a minimal realization

Note that the transfer matrix is not strictly proper. We can rewrite it in the following way:

$$R(s) = \begin{bmatrix} \frac{1}{s+1} & \frac{2}{s+1} \\ \frac{-1}{(s+1)(s+2)} & \frac{1}{s+2} \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \tilde{R}(s) + R(\infty)$$

Then, let us determine the least common denominator of $\tilde{R}(s)$:

$$\chi(s) = (s+1)(s+2) = s^2 + 3s + 2 \quad \Rightarrow r = 2, a_1 = 3, a_2 = 2$$

The Markov parameters, R_i, can be determined by:

$$\begin{aligned}
\tilde{R}_{11} &= \tfrac{1}{s+1} = s^{-1} - s^{-2} + s^{-3} + ... \\
\tilde{R}_{12} &= 2\tilde{R}_{11} \\
\tilde{R}_{21} &= \tfrac{-1}{(s+1)(s+2)} = -1(s^{-1} - s^{-2} + ...)(s^{-1} - 2s^{-2} + ...) = -s^{-2} + 3s^{-3} - 7s^{-4} + ... \\
\tilde{R}_{22} &= \tfrac{1}{s+2} = s^{-1} - 2s^{-2} + 4s^{-3} + ...
\end{aligned}$$

and so we have that:

$$\begin{aligned}
\tilde{R}(s) &= \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} s^{-1} + \begin{bmatrix} -1 & -2 \\ -1 & -2 \end{bmatrix} s^{-2} + \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} s^{-3} + \begin{bmatrix} -1 & -2 \\ -7 & -8 \end{bmatrix} s^{-4} + ... \\
&= R_1 s^{-1} + R_2 s^{-2} + R_3 s^{-3} + R_4 s^{-4} + ...
\end{aligned}$$

Besides, we have that:

$$\tilde{R}(s)\chi(s) = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} s + \begin{bmatrix} 2 & 4 \\ -1 & 1 \end{bmatrix} = N_1 s + N_0$$

a): The standard reachable form is given by:

$$\begin{aligned}
A &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 0 & -3 & 0 \\ 0 & -2 & 0 & -3 \end{bmatrix} & B &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \\
C &= \begin{bmatrix} 2 & 4 & 1 & 2 \\ -1 & 1 & 0 & 1 \end{bmatrix} & D &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}
\end{aligned}$$

b): Let us determine the McMillan degree of the system which is defined as the rank of the Hankel matrix H_2:

$$H_2 = \begin{bmatrix} R_1 & R_2 \\ R_2 & R_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -1 & -2 \\ 0 & 1 & -1 & -2 \\ -1 & -2 & 1 & 2 \\ -1 & -2 & 3 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & -2 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 2 \end{bmatrix}$$

Therefore, the McMillan degree is 3, and the previous realization is not minimal.

c): The standard observable form is given by:

$$\begin{aligned}
A &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 0 & -3 & 0 \\ 0 & -2 & 0 & -3 \end{bmatrix} & B &= \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ -1 & -2 \\ -1 & -2 \end{bmatrix} \\
C &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} & D &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}
\end{aligned}$$

d): The standard observable form is not minimal since it has dimension 4, while the McMillan degree is 3.

e): The McMillan degree is 3.

f): To determine a minimal realization we can use Ho-Kalman's algorithm, Theorem 5.3.2 in the compendium. First of all, let us determine the matrices P and Q such that:

$$PH_rQ = \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix}$$

Note that P and Q are full rank matrices which perform elementary row and column operations. So, we can determine them with the following 2 steps:

Step 1): Determine P so that:

$$PH_2 = P\begin{bmatrix} 1 & 2 & -1 & -2 \\ 0 & 1 & -1 & -2 \\ -1 & -2 & 1 & 2 \\ -1 & -2 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -1 & -2 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So, P is given by:

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

Step 2): Determine the matrix Q so that:

$$PH_2Q = \begin{bmatrix} 1 & 2 & -1 & -2 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So, Q is given by:

$$Q = \begin{bmatrix} 1 & -2 & -1 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Finally, we have all the matrices we need to apply Ho-Kalman's theorem:

$$\begin{aligned} A &= [I_n 0]P\sigma(H_2)Q\begin{bmatrix} I_n \\ 0 \end{bmatrix} = \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 \\ 1 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} -1 & -2 & 1 & 2 \\ -1 & -2 & 3 & 4 \\ 1 & 2 & -1 & -2 \\ 3 & 4 & -7 & -8 \end{bmatrix}\begin{bmatrix} 1 & -2 & -1 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \\ &= \begin{bmatrix} -1 & 0 & 0 \\ -1 & 0 & 2 \\ 1 & -1 & -3 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} B &= [I_n 0]P\begin{bmatrix} R_1 \\ R_2 \end{bmatrix} = \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 \\ 1 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ -1 & -2 \\ -1 & -2 \end{bmatrix} = \\ &= \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

$$
\begin{aligned}
C &= [R_1 R_2] Q \begin{bmatrix} I_n \\ 0 \end{bmatrix} = \\
&= \begin{bmatrix} 1 & 2 & -1 & -2 \\ 0 & 1 & -1 & -2 \end{bmatrix} \begin{bmatrix} 1 & -2 & -1 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}
\end{aligned}
$$

Therefore we have that the following system:

$$
\begin{aligned}
\dot{x} &= \begin{bmatrix} -1 & 0 & 0 \\ -1 & 0 & 2 \\ 1 & -1 & -3 \end{bmatrix} x + \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} u \\
y &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} u
\end{aligned}
$$

is a minimal realization of the transfer matrix $R(s)$.

Exercise 4.14. Let $R(s)$ be a strictly proper transfer matrix of dimension $m \times k$.

a): Assume that $m = 1$, and that R has a representation:

$$R(s) = \sum_{i=1}^{p} \frac{h_i}{s - \lambda_i}$$

where all h-vectors are nonzero, and $\lambda_i \neq \lambda_j$for $i \neq j$. Determine a minimal realization of $R(s)$.

b): Let m be arbitrary, and assume that R has a representation:

$$R(s) = \sum_{i=1}^{p} \frac{H_i}{s - \lambda_i}$$

where rank $H_i = r_i$, and $\lambda_i \neq \lambda_j$for $i \neq j$. Show that there is a realization of R of dimension $r = r_1 + r_2 + \cdots + r_p$.

Hint: If A is a linear mapping $A : \mathbb{R}^k \to \mathbb{R}^m$ with rank r, then A can be factorized as:

$$A = CB : \mathbb{R}^k \xrightarrow{B} \mathbb{R}^r \xrightarrow{C} \mathbb{R}^m$$

a): Since $R(s)$ is strictly proper we can write it as:

$$R(s) = C(sI - A)^{-1} B$$

$R(s)$ is composed of simple terms like $(s - \lambda_j)^{-1}$, so we can take A of the following form:

$$A = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_p \end{bmatrix} \Rightarrow (sI - A)^{-1} = \begin{bmatrix} (s-\lambda_1)^{-1} & & & \\ & (s-\lambda_2)^{-1} & & \\ & & \ddots & \\ & & & (s-\lambda_p)^{-1} \end{bmatrix}$$

Next, we have to find $B \in R^{p\times k}$ and $C \in R^{1\times p}$. We have that:

$$\begin{aligned}
C(sI-A)^{-1}B &= \begin{bmatrix} c_1 & ... & c_p \end{bmatrix} \begin{bmatrix} (s-\lambda_1)^{-1} & & \\ & \ddots & \\ & & (s-\lambda_p)^{-1} \end{bmatrix} \begin{bmatrix} b_{11} & ... & b_{1k} \\ \vdots & \ddots & \vdots \\ b_{p1} & ... & b_{pk} \end{bmatrix} \\
&= \begin{bmatrix} \frac{c_1}{s-\lambda_1} & ... & \frac{c_p}{s-\lambda_p} \end{bmatrix} \begin{bmatrix} b_{11} & ... & b_{1k} \\ \vdots & \ddots & \vdots \\ b_{p1} & ... & b_{pk} \end{bmatrix}
\end{aligned}$$

So, the element $(1, j)$ is:

$$R_{1,j} = \text{(from definition)} = \sum_{i=1}^{p} \frac{h_{ij}}{s-\lambda_i} = \text{(from above)} = \sum_{i=1}^{p} \frac{c_i}{s-\lambda_i} b_{ij}$$

Hence, we must choose B and C so that:

$$c_i b_{ij} = h_{ij}$$

For instance, we can take $c_i = 1$, and $b_{ij} = h_{ij}$

Note that dim $A = p$ which is also the degree of the least common denominator of $R(s)$. But this is also a lower bound for the McMillan degree of the system. Therefore, the realization is minimal.

b): Let us take inspiration from part a), and consider:

$$A = \begin{bmatrix} \lambda_1 I_{r_1} & & \\ & \ddots & \\ & & \lambda_p I_{r_p} \end{bmatrix} \Rightarrow (sI-A)^{-1} = \begin{bmatrix} (s-\lambda_1)^{-1} I_{r_1} & & \\ & \ddots & \\ & & (s-\lambda_p)^{-1} I_{r_p} \end{bmatrix}$$

By exploiting the hint, let us also consider the following factorization:

$$H_i = C_i B_i, \quad C_i \in \mathbb{R}^{m\times r_i}, B_i \in \mathbb{R}^{r_i \times k}$$

Then, we have that:

$$\begin{aligned}
R(s) &= \sum_{1=1}^{p} \frac{H_i}{s-\lambda_i} = \sum_{1=1}^{p} \frac{C_i B_i}{s-\lambda_i} \\
&= \begin{bmatrix} C_1 & ... & C_p \end{bmatrix} \begin{bmatrix} (s-\lambda_1)^{-1} I_{r_1} & & \\ & \ddots & \\ & & (s-\lambda_p)^{-1} I_{r_p} \end{bmatrix} \begin{bmatrix} B_1 \\ \vdots \\ B_p \end{bmatrix}
\end{aligned}$$

Therefore, the following:

$$\begin{aligned}
A &= \begin{bmatrix} \lambda_1 I_{r_1} & & \\ & \ddots & \\ & & \lambda_p I_{r_p} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ \vdots \\ B_p \end{bmatrix} \\
C &= \begin{bmatrix} C_1 & ... & C_p \end{bmatrix}
\end{aligned}$$

is a realization of $R(s)$ of dimension $r = r_1 + r_2 + \cdots + r_p$.