SYSTEMTEORI - ÖVNING 5: FEEDBACK, POLE ASSIGNMENT AND OBSERVER

Exercise 5.4. Consider the system:

$$\ddot{x} = a\dot{x} + bx + u$$

where u is the input and x the output signal.

(a): Determine a state space realization.
(b): Is the system reachable?
(c): Determine a state feedback so that the closed loop system has poles in $-1 \pm i$.
(d): Is it possible to stabilize the system with an output feedback $y = x$? If we take the output signal $y = \dot{x}$, is it possible to determine an output feedback to stabilize the system?
(e): Determine a regulator for the case $y = x$.

(a): To determine a state space realization, we simply take $x_1 = x$, and $x_2 = \dot{x}$. Then the system becomes:

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ b & a \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \end{aligned}$$

(b): The system is reachable since it is in the standard reachable form.
(c): We have to determine a matrix K so that the closed loop system has poles in $-1 \pm i$:

$$\text{eig}(A + BK) = -1 \pm i$$

We can use two different methods to find the right K.
(1) By substitution. The desired characteristic polynomial is:

$$\varphi(s) = (s - (-1 + i))(s - (-1 - i)) = s^2 + 2s + 2.$$

By setting $K = [k_1, k_2]$, we have that:

$$A + BK = \begin{bmatrix} 0 & 1 \\ b & a \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ b + k_1 & a + k_2 \end{bmatrix}.$$

Let us compute the characteristic polynomial of $A + BK$:

$$\det(sI - (A + BK)) = s^2 - (a + k_2)s - (b + k_1).$$

From this, we get that K has to be such that:

$$\begin{aligned} -a - k_2 = 2 \quad &\Rightarrow \quad k_2 = -2 - a, \\ -b - k_1 = 2 \quad &\Rightarrow \quad k_1 = -2 - b. \end{aligned}$$

(2) By using the standard reachable form, Lemma 6.1.4 in the compendium. Suppose that we have a completely reachable system (A, B), with $B \in \mathbb{R}^n$. Let χ_A be the characteristic polynomial of A:

$$\chi_A(s) := \det(sI - A) = s^n + a_1 s^{n-1} + \cdots + a_n.$$

Then, from Lemma 6.1.4, there is a nonsingular matrix T such that[1]:

$$TAT^{-1} = F = \begin{bmatrix} 0 & 1 & 0 & ... & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix}, \quad TB = h = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

According to Lemma 6.1.4, T is given by:

$$T = \begin{bmatrix} t \\ tA \\ \vdots \\ tA^{n-1} \end{bmatrix},$$

where the row vector t is uniquely determined by:

$$t \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 1 \end{bmatrix}.$$

Note that we do not need T to determine the standard reachable form.(We need only the characteristic polynomial χ_A.)
By writing the system in the standard reachable form, we just need to find a vector $g = [g_n, g_{n-1}, \cdots, g_1]$ such that:

$$\begin{aligned} F + hg &= \begin{bmatrix} 0 & 1 & 0 & ... & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & ... & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ g_n & g_{n-1} & g_{n-2} & ... & g_1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & 0 & ... & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ -\gamma_n & -\gamma_{n-1} & -\gamma_{n-2} & ... & -\gamma_1 \end{bmatrix} \quad \Rightarrow g_i = a_i - \gamma_i, \quad i = 1, \ldots, n, \end{aligned}$$

where γ_i are the coefficients of the desired characteristic polynomial. Once the vector g has been determined, we have to transform back the system to the original form. That is:

$$A + Bk = T^{-1}(F + hg)T.$$

Therefore, the state feedback vector k is given by:

$$k = gT.$$

[1]The matrix F is called the companion form.

Let us apply the method to this example. In this case (A, B) are already in the standard reachable form. Hence, the transformation matrix T is the identity. We can check this by applying the lemma:

$$t \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 0 & 1 \end{bmatrix} \Rightarrow t_1 = 1, t_2 = 0$$

$$\Rightarrow T = \begin{bmatrix} t \\ tA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The desired characteristic polynomial is:

$$\varphi(s) = (s - (-1 + i))(s - (-1 - i)) = s^2 + 2s + 2 \Rightarrow \gamma_1 = 2, \gamma_2 = 2$$

Then, the vector g is given by:

$$\begin{aligned} g_1 &= a_1 - \gamma_1 = -a - 2 \\ g_2 &= a_2 - \gamma_2 = -b - 2 \end{aligned}$$

Finally, we have that:

$$K = gT = \begin{bmatrix} g_2 & g_1 \end{bmatrix} I = \begin{bmatrix} -b - 2 & -a - 2 \end{bmatrix}.$$

(d): To check if it is possible to stabilize the system with an output signal feedback, we have to check if the system is both stabilizable and detectable.

The system is reachable, since it is written in the standard reachable form. Therefore, it is stabilizable. To check the detectability, we first form the observability matrix:

$$\Omega = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

which is of full rank. Therefore, the system is observable, and thus detectable. So, we can stabilize the system by output feedback.

If we take the output signal $y = \dot{x} = x_2$, then the C matrix becomes:

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow C = \begin{bmatrix} 0 & 1 \end{bmatrix}.$$

In this case the observability matrix becomes:

$$\Omega = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ b & a \end{bmatrix}$$

and it has full rank if $b \neq 0$. Hence, if $b \neq 0$, then the system is observable and detectable, and can be stabilized with output feedback.

How about the case when $b = 0$? In this case, $\text{rank}\Omega = 1$, and therefore, the system is not observable. Now, we use the decomposition theorem for observability. To this end, we change the variable $x = Tz$ with the transformation matrix

$$T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Then, the system becomes

$$\begin{aligned} \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} &= \begin{bmatrix} a & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u, \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}. \end{aligned}$$

Since the "A-matrix" of the unobservable part is not a stability matrix, the system is not asymptotically stabilizable.

(e): The design of an output feedback can be done in two separate steps: the design of a state feedback gain K, and the design of a state observer gain L. Since a controller gain K has already been designed in part (c), we have only to determine an observer. Let us choose two stable poles for the observer, for example $-2\pm i$. The desired characteristic polynomial is then:

$$\varphi(s) = (s-(-2-i))(s-(-2+i)) = s^2+4s+5 \Rightarrow \gamma_1 = 4, \gamma_2 = 5$$

Then, let us determine a feedback matrix $K = -L^T$ for the reachable system $(\tilde{A}, \tilde{b}) \triangleq (A^T, C^T)$.

The characteristic polynomial of $\tilde{A}$ is:

$$\chi_{\tilde{A}}(s) = \det(sI - \tilde{A}) = s^2 - as - b \Rightarrow a_1 = -a, a_2 = -b$$

We look for a compensator $g = [g_1, g_2]$ such that:

$$\begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ g_2 & g_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\gamma_2 & -\gamma_1 \end{bmatrix}$$

$$\Rightarrow \left\{ \begin{array}{ccccc} g_2 & = & a_2 - \gamma_2 & = & -b-5 \\ g_1 & = & a_1 - \gamma_1 & = & -a-4 \end{array} \right.$$

In this case $\tilde{A}$ is not in companion form, like before, and so we have to determine the transformation matrix T:

$$\begin{bmatrix} t_1 & t_2 \end{bmatrix} \begin{bmatrix} \tilde{b} & \tilde{A}\tilde{b} \end{bmatrix} = \begin{bmatrix} t_1 & t_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \end{bmatrix} \Rightarrow t_1 = 0, t_2 = 1$$

$$\Rightarrow T = \begin{bmatrix} t \\ t\tilde{A} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & a \end{bmatrix}$$

Then, we get that:

$$K = gT = \begin{bmatrix} -b-5 & -a-4 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & a \end{bmatrix} = \begin{bmatrix} -a-4 & -b-a^2-4a-5 \end{bmatrix}$$

Therefore, we have that:

$$L = -K^T = \begin{bmatrix} a+4 \\ a^2+4a+b+5 \end{bmatrix}$$

Together with the controller K derived in part (c), we have the following dynamical regulator[2]:

$$\frac{d}{dt} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} = \begin{bmatrix} A & BK \\ LC & A-LC+BK \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \begin{bmatrix} I \\ I \end{bmatrix} Bv.$$

[2]Note that the last term "$\begin{bmatrix} I \\ I \end{bmatrix} Bv$" is correct, and the one "$Bv \begin{bmatrix} I \\ I \end{bmatrix}$" which is shown in page 62 of the compendium is wrong.

Exercise 5.13. Consider the following system:

$$\dot{x} = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} u.$$

(a): Show that the system is unstable.
(b): Can you, using state feedback, place the poles of the close loop system in the following locations?
  (i): $-2, -2, -1, -1$.
  (ii): $-2, -2, -2, -1$.
  (iii): $-2, -2, -2, -2$.

(a): Note that the matrix A is in Jordan form. The eigenvalues are then $[-1, -1, 2, 2]$, and the Jordan block corresponding to the unstable eigenvalues is of size 2. Therefore, the system is unstable.
(b): Note that the state x_4 is not reachable, since we cannot influence it with the input u. The remaining part of the system is instead completely reachable. In fact, we have that the reachability matrix Γ has rank 3:

$$\Gamma = [B, AB, A^2B, A^3B] = \begin{bmatrix} 0 & 1 & 4 & 12 \\ 1 & 2 & 4 & 8 \\ 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Note also that the system is already in the reachable form.

Therefore, by using state feedback, we can place the poles of the close loop system freely, except one pole at -1 cannot be moved. Thus, the pole placement is possible for the locations (i) and (ii), but not (iii).

Exercise 5.10. Consider the following discrete-time system:

$$x(k+1) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -3 & -4 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} u(k).$$

Find a linear feedback such that for any initial state, the system returns to the origin in a fixed number of steps (dead-beat control).

The closed loop system is:

$$x(k+1) = (A + BK)x(k).$$

The solution of the previous system is:

$$x(k+N) = (A + BK)^N x(k)$$

We have a dead-beat controller if

$$(A + BK)^N = 0$$

for some $N > 0$.

We can achieve such result by using Cayley-Hamilton theorem. If the characteristic polynomial of the closed loop system is $\chi_{A+BK}(z) = z^3$, then we have that $(A + BK)^3 = 0$. So what we need to do is to place all the poles of the system in the origin.

We use the pole assignment algorithm. First, we compute

$$t\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & -5 \\ 0 & -5 & 18 \end{bmatrix} = [0,0,1] \quad \Rightarrow \quad t = \frac{1}{43}\left[5,-5,1\right].$$

Hence, we can obtain the transformation matrix T as

$$T = \begin{bmatrix} t \\ tA \\ tA^2 \end{bmatrix} = \frac{1}{43}\begin{bmatrix} 5 & -5 & 1 \\ -2 & 2 & -9 \\ 18 & 25 & 38 \end{bmatrix}.$$

Since $g = [2,3,4]$, we can obtain the state feedback matrix as

$$K = gT = \frac{1}{43}[76, 96, 127].$$

Exercise. Prove the following theorem:
Theorem: Let $\Sigma = (A, B, C)$ a LTI system in a reachable form, that is:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix},$$

where (A_{11}, B_1) is completely reachable. Consider the feedback system $\Sigma_K = (A+BK, B, C)$. Σ_K is also in a reachable form, and has the same unreachable part A_{22}.

Let us partition the feedback gain matrix K accordingly to B as $[K_1, K_2]$. Then, it is easy to show that:

$$A + BK = \begin{bmatrix} A_{11} + B_1K_1 & A_{12} + B_1K_2 \\ 0 & A_{22} \end{bmatrix}.$$

Therefore, the unreachable poles are not affected by the state feedback. So, in order to analyze the effects of feedback, it is enough to consider only the reachable part of the system. If A_{22} is a stability matrix, then the pair (A, B) is called **stabilizable**[3]. As a dual concept of stabilizability, the pair (A, C) is called **detectable** if (A', C') is stabilizable.

Exercise from Tenta 031022. Let us consider the following LTI system:

$$\dot{x} = \begin{bmatrix} 13 & 6 & -8 \\ -12 & -5 & 8 \\ 8 & 4 & -5 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u.$$

Find an appropriate control input $u = Kx$ to stabilize the system.

First of all, let us compute the poles of the system. The characteristic polynomial of A is:

$$\chi_A(\lambda) = \det(\lambda I - A) = (\lambda - 3)(\lambda - 1)(\lambda + 1)$$

[3] In other words, a system is stabilizable if its unstable subspace is contained in its reachable subspace.

and hence the system is unstable. To check if it is possible to stabilize it, we have to determine its reachable space:

$$\Gamma = [B, AB, A^2B] = \begin{bmatrix} 1 & 5 & 17 \\ 0 & -4 & -16 \\ 1 & 3 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & 0 \\ 0 & -4 & 0 \\ 1 & 3 & 0 \end{bmatrix}$$

So, the reachable space has dimension 2 and hence the system is not completely reachable. Is it stabilizable? Let us transform the system to a reachable form using the following transformation matrix:

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

In the new basis the system has the following form:

$$\dot{z} = T^{-1}ATz + T^{-1}Bu = \begin{bmatrix} 5 & 4 & -8 \\ -2 & -1 & 4 \\ 0 & 0 & -1 \end{bmatrix} z + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$

which is clearly stabilizable.

In order to determine a stabilizing feedback, we can consider only the reachable subspace:

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

Let $u = \alpha_1 z_1 + \alpha_2 z_2 + v$ and take the new poles in $[-2, -3]$:

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} 5+\alpha_1 & 4+\alpha_2 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} v$$

The parameters α_1 and α_2 have to be such that:

$$\chi(\lambda) = \lambda^2 - (4+\alpha_1)\lambda + 3 - \alpha_1 + 2\alpha_2 = (\lambda+2)(\lambda+3) \Leftrightarrow \alpha_1 = -9, \alpha_2 = -3$$

Therefore, for the overall system we can take $u = [-9, -3, 0]z + v$ so that:

$$\dot{z} = T^{-1}ATz + T^{-1}Bu = \begin{bmatrix} -4 & 1 & -8 \\ -2 & -1 & 4 \\ 0 & 0 & -1 \end{bmatrix} z + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} v$$

Finally, since $z = T^{-1}x$ we have that the control input in the original coordinate system is:

$$u = -[9, 3, 0] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ -1 & -1/2 & 1 \end{bmatrix} x + v = -[9, 3/2, 0]x + v.$$

Example (State feedback with multiple inputs). Suppose that a linear (unstable) system is described by

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} u.$$

This system is reachable; indeed it is in the standard reachable form in a multiple input case. Design a state feedback gain matrix K that places all the closed-loop poles at -1.

The desired characteristic polynomial is

$$\chi_{A+BK}(s) = (s+1)^4 = s^4 + 4s^3 + 6s^2 + 4s + 1.$$

So, we aim at obtaining a closed-loop system matrix of the companion matrix form:

$$A + BK = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -4 & -6 & -4 \end{bmatrix}. \tag{0.1}$$

Suppose that

$$K = \begin{bmatrix} k_1 & k_2 & k_3 & k_4 \\ k_5 & k_6 & k_7 & k_8 \end{bmatrix}.$$

Then, the closed-loop system matrix becomes

$$A + BK = \begin{bmatrix} 0 & 1 & 0 & 0 \\ k_1+1 & k_2+1 & k_3+1 & k_4+1 \\ 0 & 0 & 0 & 1 \\ k_5+1 & k_6+1 & k_7+1 & k_8+1 \end{bmatrix}. \tag{0.2}$$

By comparing the elements in (0.1) and (0.2), we can obtain

$$K = \begin{bmatrix} -1 & -1 & 0 & -1 \\ -2 & -5 & -7 & -5 \end{bmatrix}.$$