SYSTEMTEORI - ÖVNING 6: LINEAR-QUADRATIC OPTIMAL CONTROL

1. Examples

1.1. **6.1.** Determine the control law $u(t, x)$ minimizing

$$J(u) = \frac{1}{2}\int_0^T (3x^2 + \frac{1}{4}u^2)dt + x(T)^2$$

under the constraint:

$$\dot{x} = 2x + u$$

The general cost function for LQ problems is the following:

$$J(u) = \int_0^T (x'Qx + u'Ru)dt + x(T)'Sx(T)$$

where $Q \geq 0$, $S \geq 0$, and $R > 0$. In this case, we have a SISO problem with:

$$Q = \frac{3}{2}, \quad R = \frac{1}{8}, \quad S = 1$$

It is well known that the optimal solution, $\hat{u}(t)$, of this kind of problems has the form of a linear state feedback and is expressed by:

$$\hat{u}(t) = -R^{-1}BP(t)x(t) = -8P(t)x(t)$$

where $P(t)$ is the unique positive semidefinite solution of the Riccati equation (RE):

$$\begin{aligned} \dot{P} &= -A^T P - PA + PBR^{-1}B^T P - Q \\ P(T) &= S \end{aligned}$$

which is positive semi-definite and bounded. In this case we have:

$$\begin{aligned} \dot{P} &= -4P + 8P^2 - \frac{3}{2} \\ P(T) &= 1 \end{aligned}$$

which is a non-linear ODE, and therefore quite difficult to solve. Next, we will see two ways of solving it:

(1) In the general multidimensional case, it is easier to solve the RE by using the method described in section 7.2 of the compendium. So, by defining $P = YX^{-1}$ we get:

$$\begin{aligned} \dot{X} &= AX - BR^{-1}B^T Y, \quad X(T) = I \\ \dot{Y} &= -QX - A^T Y, \qquad Y(T) = S \end{aligned}$$

In this case we get the following linear system:

$$\begin{aligned} \dot{X} &= 2X - 8Y, \quad X(T) = 1 \\ \dot{Y} &= -\frac{3}{2}X - 2Y, \quad Y(T) = 1 \end{aligned}$$

That is:
$$\frac{d}{dt}\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 2 & -8 \\ -3/2 & -2 \end{bmatrix}\begin{bmatrix} X \\ Y \end{bmatrix} = A\begin{bmatrix} X \\ Y \end{bmatrix}, \quad \begin{bmatrix} X(T) \\ Y(T) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
The solution of this system is given by:
$$\begin{bmatrix} X(t) \\ Y(t) \end{bmatrix} = e^{At}\mathbf{X}_0 \text{ where } \mathbf{X}_0 = \begin{bmatrix} X(0) \\ Y(0) \end{bmatrix}$$
In order to determine the matrix exponential e^{At}, we can use Cayley-Hamilton method. The eigenvalues of A are given by:
$$\det(\lambda I - A) = \lambda^2 - 16 \Rightarrow \lambda_{1,2} = \pm 4$$
Then, we have to determine the coefficients β such that
$$e^{At} = \beta_0 I + \beta_1 A$$
We have that:
$$\begin{aligned} e^{4t} &= \beta_0 + 4\beta_1 \\ e^{-4t} &= \beta_0 - 4\beta_1 \end{aligned}$$
from which we get:
$$\begin{aligned} \beta_0 &= \frac{e^{4t} + e^{-4t}}{2} \\ \beta_1 &= \frac{e^{4t} - e^{-4t}}{8} \end{aligned}$$
and so:
$$e^{At} = \begin{bmatrix} (3e^{4t} + e^{-4t})/4 & -e^{4t} + e^{-4t} \\ (-3e^{4t} + 3e^{-4t})/16 & (e^{4t} + 3e^{-4t})/4 \end{bmatrix}$$
In this case we need also to determine the initial condition $\mathbf{X}_0$. Since we know the end boundary condition $[X(T), Y(T)]$, we can use the following equation:
$$e^{AT}\mathbf{X}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \mathbf{X}_0 = e^{-AT}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 5e^{4T} - e^{-4T} \\ (15e^{4T} - e^{-4T})/4 \end{bmatrix}$$
The solution is then:
$$\begin{bmatrix} X(t) \\ Y(t) \end{bmatrix} = e^{At}x_0 = e^{4(t-T)}\begin{bmatrix} 4(-e^{8T} + 5e^{-8t} \\ e^{-8T} + 15e^{-8T} \end{bmatrix}$$
and finally:
$$P = YX^{-1} = \frac{e^{4(t-T)}}{e^{4(t-T)}}\frac{e^{-8T} + 15e^{-8T}}{4(-e^{8T} + 5e^{-8t}} = -\frac{1}{4}\frac{1 + 15e^{8(T-t)}}{1 - 5e^{8(T-t)}}$$

(2) Another way of solving the differential equation is by using substitution. Suppose
$$P = k\frac{\dot{z}}{z}$$
for some value of the parameter k. Then, we have that:
$$\dot{P} = k\frac{\ddot{z}}{z} - k\frac{\dot{z}}{z^2}\dot{z} = -4k\frac{\dot{z}}{z} + 8k^2\frac{\dot{z}^2}{z^2} - \frac{3}{2} \Rightarrow$$
$$\Rightarrow k\frac{\ddot{z}}{z} + (-k - 8k^2)\frac{\dot{z}^2}{z^2} + 4k\frac{\dot{z}}{z} + \frac{3}{2} = 0$$

Then, we can get rid of the nonlinearity by setting $-k - 8k^2 = 0$, that is $k = -1/8$. Therefore, we get the following homogeneous differential equation:

$$\ddot{z} + 4\dot{z} - 12z = 0$$

with characteristic equation:

$$r^2 + 4r + 12 = 0 \Rightarrow r_{1,2} = 2, -6$$

Hence, the general solution of the homogeneous differential equation is:

$$\begin{aligned} z &= c_1 e^{2t} + c_2 e^{-6t} \\ \dot{z} &= 2c_1 e^{2t} - 6c_2 e^{-6t} \end{aligned}$$

where the parameters c_1 and c_2 have to be determined from the boundary conditions. In this case we have that:

$$P(t) = k\frac{\dot{z}}{z} = -\frac{1}{8}\frac{2c_1e^{2t} - 6c_2e^{-6t}}{c_1e^{2t} + c_2e^{-6t}} = -\frac{1}{4}\frac{1 - 3ce^{-8t}}{1 + ce^{-8t}}$$

where $c = c_2/c_1$. By using the boundary condition on $P(T)$ we get that:

$$P(T) = 1 \Rightarrow c = -5e^{8T}$$

Finally, we have that:

$$P(t) = -\frac{1}{4}\frac{1 + 15e^{8(T-t)}}{1 - 5e^{8(T-t)}}$$

The optimal control law is then:

$$\hat{u}(t) = -R^{-1}B'P(t)x(t) = 2\frac{1 + 15e^{8(T-t)}}{1 - 5e^{8(T-t)}}x(t)$$

which is linear in the variable x, but with a time dependent coefficient.

1.2. **6.8.** Consider the problem of minimizing

$$\int_0^{\infty} (qx^2 + ru^2)dt$$

subject to

$$\dot{x} = ax + bu \quad a, b \in \mathbb{R} \quad b \neq 0, r > 0, q > 0$$

We assume $b > 0$.

a: Determine the optimal control law
b: What happens if $b = 0$ and $a > 0$?
c: What happens if $q = 0$?
d: What happens with the gain k when $r \to \infty$? Determine the corresponding closed-loop system.

a: Let us re-write the problem in the same form as the one in pag. 72 of the compendium. By defining the variables:

$$\begin{aligned} y &= \sqrt{q}x \\ v &= \sqrt{r}u \end{aligned}$$

we get that the minimization problem becomes:

$$\int_0^\infty (\|y\|^2 + \|v\|^2)dt$$

subject to

$$\begin{aligned} \dot{x} &= ax + \frac{b}{\sqrt{r}}v \\ y &= \sqrt{q}x \end{aligned}$$

This is a minimal realization if $\frac{b}{\sqrt{r}} \neq 0$ and $\sqrt{q} \neq 0$. Then, Theorem 8.2.4 gives us the optimal control law:

$$\hat{v} = -B'\bar{P}x$$

where $\bar{P}$ is the only positive definite solution of the algebraic Riccati equation (ARE):

$$A^T P + PA - PBB'P + C'C = 0 \quad P > 0$$

In our case we get:

$$2ap - \frac{b^2p^2}{r} + q = 0 \Rightarrow p^2 - \frac{2ar}{b^2}p - \frac{rq}{b^2} = 0 \Rightarrow$$

$$\Rightarrow (p - \frac{ar}{b^2})^2 = \frac{rq}{b^2} + \frac{a^2r^2}{b^4} \Rightarrow \bar{p} = \frac{ar + \sqrt{rqb^2 + a^2r^2}}{b^2}$$

Then, we have that the optimal control law $\hat{u}$ is:

$$\hat{u}(t) = \frac{1}{\sqrt{r}}\hat{v} = \frac{1}{\sqrt{r}}(-\frac{b}{\sqrt{r}})\frac{r(a + \sqrt{qb^2/r + a^2}}{b^2} = -\frac{1}{b}\Big(a + \sqrt{a^2 + \frac{q}{r}b^2}\Big)x(t) = Kx(t)$$

Note that the feedback law depends on the ration q/r.

b: If we let $b \to 0$, we have that the feedback gain becomes:

$$K = -\frac{1}{b}(a + \sqrt{a^2 + \frac{q}{r}b^2}) = -\frac{a}{b} - \sqrt{\frac{a^2}{b^2} + \frac{q}{r}} \underset{b\to 0}{=} \begin{cases} 0 & \text{if } a < 0 \\ \infty & \text{if } a > 0 \end{cases}$$

Note that if $b = 0$ the system is not reachable, and the problem is wrong formulated. Note also that if $a < 0$ and $b = 0$ the system is stable by itself and the solution of the problem is trivially solved by $\hat{u} = 0$.

c: If $q = 0$ then the system is not observable, and the problem does not satisfy the conditions of the theory presented in the course. From the formula above we get

$$K = \frac{1}{b}(-a - \sqrt{a^2}) = \begin{cases} 0, & a < 0 \\ -\frac{2a}{b}, & a > 0 \end{cases}$$

d: If $r \to \infty$ we have that:

$$K = -\frac{a}{b} - \sqrt{\frac{a^2}{b^2} + \frac{q}{r}} \underset{r\to\infty}{=} \frac{-a - |a|}{b} = \begin{cases} 0 & \text{if } a < 0 \\ -2a/b & \text{if } a > 0 \end{cases}$$

and the closed loop system becomes:

$$\dot{x} = -|a|x$$

1.3. **Es. 3.** Consider the following differential equation:

$$\ddot{x} = u \quad x(0) = \xi_1, \dot{x}(0) = \xi_2$$

with the following cost function:

$$J(u) = \int_0^T (x^2 + u^2) dt$$

Let $T \to \infty$ and determine the optimal control law.

By setting $x_1 = x$ and $x_2 = \dot{x}$, we get the following system:

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u = Ax + Bu \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} x = Cx \end{aligned}$$

In order for the optimal solution to exist, the system has to be a minimal realization:

$$\begin{aligned} \Gamma &= \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ \Omega &= \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

Both Γ and Ω have full rank, and so the system is completely reachable and observable.

The optimal control law is given by:

$$\hat{u} = -B'\bar{P}x$$

where $\bar{P}$ is the only positive definite solution of the algebraic Riccati equation (ARE):

$$A^T P + PA - PBB'P + C'C = 0 \quad P > 0$$

In our case, by defining a matrix P in the following way:

$$P = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}$$

we get the following equations:

$$\begin{aligned} p_1^2 - 1 &= 0 \\ -p_1 + p_2 p_3 &= 0 \\ p_3^2 - 2p_2 &= 0 \end{aligned}$$

$$\Rightarrow p_2 = \pm 1 \Rightarrow \begin{cases} p_2 = 1 & \Rightarrow \quad p_1 = p_3 = \pm\sqrt{2} \\ p_2 = -1 & \Rightarrow \quad p_1 = -p_3 = \pm i\sqrt{2} \end{cases}$$

Since we have to take the positive definite solution, we have that:

$$\bar{P} = \begin{bmatrix} \sqrt{2} & 1 \\ 1 & \sqrt{2} \end{bmatrix}$$

The closed loop system is given by:

$$\dot{x} = (A - BB^T\bar{P})x = \begin{bmatrix} 0 & 1 \\ -1 & -\sqrt{2} \end{bmatrix} x$$

According to Corollary 8.1.3 the closed loop system is asymptotically stable. We can check it by computing the poles of the system:

$$\det(\lambda I - \begin{bmatrix} 0 & 1 \\ -1 & -\sqrt{2} \end{bmatrix}) = \lambda^2 - \sqrt{2} + 1 \Rightarrow \lambda_{1,2} = -\frac{1}{\sqrt{2}} \pm \frac{i}{\sqrt{2}}$$

Hence, the system is stable.

1.4. **Ex. 2 from Tenta 120104.** Determine the optimal control that minimizes the following integral:

$$\int_0^\infty (y^2 + u^2)dt$$

for the system:

$$\begin{aligned} \dot{x} &= \begin{bmatrix} -1/2 & 0 \\ 1 & -1/2 \end{bmatrix} x + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u \\ y &= \begin{bmatrix} 0 & 1 \end{bmatrix} x \end{aligned}$$

First of all, let us check if the system is minimal. We have that:

$$\Gamma = [B, AB] = \begin{bmatrix} 1 & -1/2 \\ -1 & 3/2 \end{bmatrix} \Rightarrow \text{full rank}$$

and

$$\Omega = [C, C]^T = \begin{bmatrix} 0 & 1 \\ 1 & -1/2 \end{bmatrix} \Rightarrow \text{full rank}$$

Hence the system is minimal. From theorem 8.2.4 we know that the corresponding ARE has an unique real positive definite solution such that the control input $u = -B'Px$ minimizes the cost function, and the closed loop system is asymptotically stable.

In order to determine P we have to solve the ARE:

$$-A^T P - PA + PBB^T P - CC^T = 0 \Leftrightarrow$$

After substituting the values we get the following system of equations:

$$\begin{aligned} p_{11}/2 - p_{12} + p_{11}/2 - p_{12} + (p_{11} - p_{12})^2 &= 0 \\ p_{12}/2 - p_{22} + p_{12}/2 - p_{22} + (p_{11} - p_{12})(p_{12} - p_{22}) &= 0 \\ p_{22}/2 + p_{22}/2 + (p_{12} - p_{22})^2 - 1 &= 0 \end{aligned}$$

and then

$$\begin{aligned} p_{11} - 2p_{12} + (p_{11} - p_{12})^2 &= 0 \\ (p_{12} - p_{22})(1 + p_{11} - p_{12}) &= 0 \\ p_{22} + (p_{12} - p_{22})^2 - 1 &= 0 \end{aligned}$$

The second equation has 2 solutions: $p_{12} - p_{22} = 0$ and $1 + p_{11} - p_{12} = 0$. It is easy to see that the second one is not allowed. Finally we get that:

$$P = \begin{bmatrix} (1+\sqrt{5})/2) & 1 \\ 1 & 1 \end{bmatrix}$$

The optimal control law is then:

$$u = -B^T Px = -\frac{1}{2}(\sqrt{5} - 1)x_1$$

1.5. **Es. 4.** Suppose we have a dc motor which has a shaft with angular velocity $\xi(t)$ and which is driven by the input voltage $\mu(t)$. The system is described by the following scalar state differential equation:

$$\dot{\xi}(t) = -\alpha\xi(t) + \beta\mu(t)$$

with α and β given constants. We want to stabilize the angular velocity $\xi(t)$ at a desired value ω_0. Let μ_0 be the constant input voltage to which ω_0 corresponds as the steady-state angular velocity, that is $-\alpha\xi_0 + \beta\mu_0 = 0$. As the optimization criterium we choose the following:

$$\int_0^T ((\xi(t) - \xi_0)^2 + r(\mu(t) - \mu_0)^2)dt + s(\xi(T) - \xi_0)^2$$

Find the optimal control law for $r = 1/8$ and $s = 1$. What happens if $T \to \infty$?

First of all we have to shift the origin, since in our formulation of the optimal regulator problem we have set the origin of the state space as the equilibrium point. Therefore, let us introduce the new variables:

$$\begin{aligned} x(t) &= (\xi(t) - \xi_0)/\beta \\ u(t) &= (\mu(t) - \mu_0)/(-\alpha) \end{aligned}$$

With these new variables the problem becomes:

$$\min_u \int_0^T (x^2 + \frac{u^2}{8})dt + x(T)^2$$

subjected to:

$$\dot{x} = x + u$$

Note that the cost function ensures that $\xi(t)$ stays close to ω_0, that $\mu(t)$ does not deviate too much from μ_0 and that the terminal state $\xi(T)$ will be closed to ω_0. In general, the values of the parameters r and s has to be determined by trial and error.

The optimal control law is given by:

$$u(t) = -8p(t)x(t)$$

where $p(t)$ is the unique solution of the RE:

$$\begin{aligned} \dot{p} &= -2p + 8p^2 - 1 \\ p(T) &= 1 \end{aligned}$$

We can solve the previous differential equation using one of the two methods described in the first example. The solution is:

$$p(t) = -\frac{1}{2}\frac{5 + e^{6(t-T)}}{-5 + 2e^{6(t-T)}}$$

and so the optimal control law is:

$$u(t) = 8\frac{1}{2}\frac{5 + e^{6(t-T)}}{-5 + 2e^{6(t-T)}}x(t)$$

If we suppose that the final time T grows to infinity we have that:

$$p(t) \to \bar{p} = \frac{1}{2}, \quad \Rightarrow \bar{u}(t) = -4x$$

and the closed loop system becomes:

$$\dot{x}(t) = -3x(t)$$

which is clearly asymptotically stable.

We can also determine $\bar{p}$ by solving the ARE as described in Chapter 8 of the compendium. However, we need to formulate the problem in a slightly different way: minimize the cost function:

$$J(v) = \int_0^\infty (\|y\|^2 + \|v\|^2)dt$$

subjected to:

$$\begin{aligned} \dot{x} &= x + \sqrt{8}v \\ y &= x \end{aligned}$$

where we introduced the new input $v = u/\sqrt{8}$. The system is reachable and observable, and so it is a minimal realization. Theorem 8.2.4 tells us that the unique optimal control that minimizes $J(v)$ and stabilizes the system is given by:

$$\bar{v} = -\sqrt{8}\bar{p}x$$

where $\bar{p}$ is the unique positive definite solution of the ARE. In this case:

$$\text{(ARE)} \qquad 2p - 8p^2 + 1 = 0 \Rightarrow p_{1,2} = \frac{1}{2}, -\frac{1}{4} \quad \Rightarrow \bar{p} = \frac{1}{2}$$

and so the optimal input $\bar{v}$ is:

$$\bar{v} = -\sqrt{8}\bar{p}x = -\frac{\sqrt{8}}{2}x \Rightarrow \bar{u} = -4x$$

which corresponds to the previously computed optimal solution.