KTH Matematik

Exam October 24, 2007 in SF2832 Mathematical Systems Theory.

Examiner: Ulf Jönsson, tel. 790 84 50.

Allowed books: Anders Lindquist & Janne Sand, An Introduction to Mathematical Systems Theory and β mathematics handbook.

Solution methods: All conclusions should be carefully motivated.

Note! Your personal number must be stated on the cover sheet. Number your pages and write your name on each sheet that you turn in!

You need 45 points credit (including your bonus) to pass the exam.

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1. Consider the following system:

$$\dot{x}(t) = \begin{pmatrix} -1 & -3 \\ -3 & -1 \end{pmatrix} x(t) + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u(t).$$

(a) Is the system completely reachable? . . . . (4p)

(b) Determine the transition function $\Phi(t,s)$ of the system. . . . . (8p)

(c) Let $x(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and let $u(t) = 1$ for $t \geq 0$. Determine $x(t)$ for $t \geq 0$. What happens when $t \to \infty$. . . . . (3p)

(d) Let $x(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and let $u(t) = 1$ for $t \geq 0$. Determine $x(t)$ for $t \geq 0$. What happens when $t \to \infty$. . . . . (3p)

(e) Let $x(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Find a $u(t)$ such that $x(t) \to 0$ as $t \to \infty$.

Do not spend to much time on this if you don't see the answer right away. . . . . (2p)

2. Two pendulums, coupled by a spring, are to be controlled by two equal and opposite forces u which are applied to the pendulum bobs. See Figure 1 for an illustration. When θ_1 and θ_2 are small, the governing equations of the system can be well approximated by the following second order differential equations:

$$\begin{aligned} m_1 l_1^2 \ddot{\theta}_1 &= -ka^2(\theta_1 - \theta_2) - m_1 g l_1 \theta_1 - l_1 u \\ m_2 l_2^2 \ddot{\theta}_2 &= -ka^2(\theta_2 - \theta_1) - m_2 g l_2 \theta_2 + l_2 u \end{aligned} \quad (1)$$

where g denotes the acceleration of the gravity.

(a) Let $x = \begin{bmatrix} \theta_1 & \theta_2 & \dot{\theta}_1 & \dot{\theta}_2 \end{bmatrix}^T$. Derive a state space model from (1)

$$\dot{x} = Ax + Bu$$

in the case when $m_1 = m_2 = 0.04$, $a = 1$, $l_1 = l_2 = 5$, $k = 1$, and $g = 10$. . (5p)

Figur 1: The two-pendulum system.

(b) Is the system in (a) completely reachable? .... (5p)

(c) Is the system in (a) completely observable with measurement $y = \theta_1$? ... (5p)

(d) Let $z = \begin{bmatrix}\theta_1 + \theta_2 & \dot{\theta}_1 + \dot{\theta}_2 & \theta_1 - \theta_2 & \dot{\theta}_1 - \dot{\theta}_2\end{bmatrix}$. Derive a state space model for (1) using the new state z. Identify those elements of z which are reachable and those which are not reachable. .... (5p)

3. Consider the cost function

$$J[u] = \int_0^\infty (x^TQx + u^2)dt \quad \text{subj. to} \quad \dot{x} = Ax + Bu,\ x(0) = x_0$$

where

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad x_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

(a) Compute $J[0]$. .... (10p)

(b) Compute $\min_{u \in \mathcal{U}} J[u]$, where

$$\mathcal{U} = \left\{ u : \int_0^\infty (x^Tx + u^2)dt < \infty \right\}$$

.... (10p)

4. Answer the following true-false questions. You must justify your answers.

(a) Given a strictly proper rational transfer matrix $R(s)$ such that

$$\begin{aligned} R(s) &= \frac{1}{\chi(s)} \left(N_0 + N_1 s + \ldots N_{r-1} s^{r-1} \right) \\ &= R_1 s^{-1} + R_2 s^{-2} + R_3 s^{-3} \ldots \end{aligned}$$

where $\chi(s) = s^r + a_1 s^{r-1} + \ldots + a_r$ is the least common denominator of the elements $R_{ij}(s)$ of $R(s)$). Suppose

$$\begin{aligned} &\deg(\chi(s)) = 2 \\ &R_k = (-1)^k, \quad k = 1, 2, 3, \ldots \end{aligned}$$

True or False: The McMillan degree of $R(s)$ is 2. .... (5p)

(b) True or False: It is possible to place the eigenvalues of the following system at arbitrary location in the complex plane by using an appropriate state feedback $u = Kx$:

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} u$$

.... (5p)

(c) Consider a system $G : u \to y$ where the input output relationship is characterized by the equation:

$$y(t) = \int_{-\infty}^{\infty} h(t-s)u(s-T)ds$$

where $h(t)$ is a given function and T is a given constant. We assume $h(t) \neq 0$ for all t. True or False:

(i) G is a linear system. .... (2p)

(ii) G is a time-invariant system. .... (2p)

(iii) G is a causal system. .... (1p)

(d) Let matrices A, B, and C be $n \times n$, $n \times p$, and $m \times n$, respectively. Suppose a non-zero eigenvector, v, of matrix A satisfies $Cv = 0$. True or False: the system

$$\begin{aligned} \dot{z} &= (A + BKC)z + Bw, \\ y &= Cz \end{aligned}$$

is not completely observable *for any* $p \times m$ matrix K. .... (5p)

5. In this problem we investigate the relative degree and zeros of single-input-single-output systems (SISO). Consider the state space representation

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned}$$

and its correponding transfer function

$$G(s) = C(sI - A)^{-1}B = \frac{b_m s^m + b_{m-1}s^{m-1} + \ldots + b_0}{s^n + a_1 s^{n-1} + \ldots + a_n}$$

where $A \in \mathbf{R}^{n\times n}$, $B \in \mathbf{R}^{n\times 1}$, $C \in \mathbf{R}^{1\times n}$.

(a) The relative degree $d = n-m$ is the excess degree of the denominator compared to the numerator. Prove that

$$CA^kB = 0, \quad k = 0, 1, \ldots, d-2 \quad \text{and} \quad CA^{d-1}B \neq 0$$

.... (8p)

(b) We may factorize the transfer function using the poles and zeros

$$G(s) = C(sI - A)^{-1}B = b_m \frac{(s-z_1)(s-z_2)\cdot \ldots \ldots \cdot (s-z_m)}{(s-p_1)(s-p_2)\cdot \ldots \cdot (s-p_n)}$$

To find an intepretation of the zeros we use that a complex exponential $u(t) = \bar{u}e^{st}$ in stationarity gives rise to the complex exponential output $y(t) = G(s)\bar{u}e^{st}$. Hence, if the frequency is equal to either of the zeros, $s = z_k$, $k \in \{1, \ldots, m\}$, then the stationary output is zero since $y(t) = G(z_k)\bar{u}e^{z_k t} = 0$. Prove that the zeros can be characterized as those complex numbers s such that

$$\text{rank}\left(\begin{bmatrix} -sI + A & B \\ C & 0 \end{bmatrix}\right) < n + 1$$

Hint: Use that the stationary state space solution has the form $x(t) = \bar{x}e^{st}$. (8p)

(c) Consider the case when

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad C = \begin{bmatrix} -2 & 1 \end{bmatrix}$$

Use the above characterizations to verify that $G(s) = C(sI - A)^{-1}B$ has the relative degree $d = 1$ and a zero at $z = 2$. . . . . (4p)

Good luck!