KTH Matematik

Exam October 24, 2008 in SF2832 Mathematical Systems Theory.

Examiner: Xiaoming Hu, tel. 790 7180.

Allowed books: Anders Lindquist & Janne Sand, An Introduction to Mathematical Systems Theory, class notes and β mathematics handbook.

Solution methods: All conclusions should be carefully motivated.

Note! Your personal number must be stated on the cover sheet. Number your pages and write your name on each sheet that you turn in!

You need 45 points credit (including your bonus) to pass the exam.

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1. Determine if each of the following statements is true or false. You must justify your answers.

(a) Consider $\dot{x} = Ax$ and assume $A^T = -A$. True or False: $x(t)$ always moves in a circle as long as $x(0) \neq 0$. .... (5p)

(b) True or False: Given a minimal realization (A, B, C), $(A + BLC, B, C)$ is also minimal with any L. .... (5p)

(c) Consider $\dot{x} = Ax$. Assume the equation $Ax = 0$ has more than one solution. True or False:

(i) The system may be asymptotically stable. .... (3p)

(ii) There exists an initial condition $x_0 \neq 0$ and a finite $T \geq 0$, such that $e^{At}x_0 = 0, \ \forall t \geq T$. .... (2p)

(d) Consider a controllable and observable system

$$\dot{x} = Ax + bu,$$
$$y = Cx,$$

where $x \in R^n$, $u \in R$ and $y \in R^p$, and $p < n$. True or False: arbitrary pole assignment is never possible with output feedback $u = Ly$. .... (5p)

2. Consider :

$$\dot{x} = \begin{pmatrix} -(1+a) & -a \\ 1 & 0 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u$$
$$y = \begin{pmatrix} 0 & 1 \end{pmatrix} x.$$

(a) Is the system minimal? .... (6p)

(b) Determine the state transition matrix e^{At} of the system. .... (8p)

(c) Discuss when the system is BIBO stable. ....(6p)

3. Consider the linearized system of attitude control for a spacecraft:

$$\begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}$$

$$\begin{pmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3 \end{pmatrix} = \sum_{i=1}^{m} b_i u_i,$$

where ϕ, θ, ψ describe the angular position of the spacecraft, and ω_i are angular velocities. Each control u_i is implemented by an opposing pair of gas jets. b_i depends on where the gas jets are placed.

(a) At least how many controls are needed in order to make the system controllable? (6p)

(b) At least how many measurements are needed in order to make the system observable? ....(6p)

(c) Design a set of b_i such that the system is controllable and design a state feedback control so that the poles of the closed-loop system are placed at $\{-1, -1, -1, -1, -1, -1\}$. ....(8p)

4. Consider the optimal control problem for a controllable and observable system

$$\min_u J = \int_0^{\infty} (x^T Q x + u^2) dt \quad \text{s.t.} \quad \dot{x} = Ax + Bu, \ x \in R^n, \ u \in R^m, \ x(0) = x_0.$$

Suppose P^* is the positive definite solution for the associated algebraic Riccati equation.

(a) Show the following linear matrix inequality (LMI) holds:

$$\begin{bmatrix} A^T P^* + P^* A + Q & P^* B \\ B^T P^* & I \end{bmatrix} \geq 0.$$

....(8p)

(b) Define

$$F(P) = \begin{bmatrix} A^T P + PA + Q & PB \\ B^T P & I \end{bmatrix},$$

show for any real symmetric matrix P,

$$\text{rank} F(P) = m + \text{rank}(A^T P + PA + Q - PBB^T P)$$

(which implies that P^* minimizes the rank of $F(P)$.) ....(5p)

(c) Now let $A = a > 0, \ B = 1, \ Q = q$, compute the optimal control $u = kx$. (5p) (*d*) What is the closed-loop pole as $q \to 0$? .... (2p)

5. Let z be the outcome of a random variable with distribution $N(0, \alpha^2)$ (i.e., $E\{z\} = 0$, $E\{z^2\} = \alpha^2$). We would like to determine the value of z by a set of noisy measurements

$$y(t) = z + w(t) \text{ for } t = 0, 1, \ldots, n-1$$

where $w(t) \in N(0, \sigma^2)$ are independent of each other and of z.

a) Determine $P(n) = E\{(z - \hat{z}_n)^2\}$ as a function of α, σ, n, where $\hat{z}_n$ is the optimal estimate of z based on measurements up to time instant $n-1$. (*Hint:* note that $f(k+1)^{-1} = f(k)^{-1} + \delta \Rightarrow f(k) = \frac{1}{f(0)^{-1} + k\delta}$.) (8p)

b) Find $\hat{z}_n$ of z in terms of $y(0), y(1), \ldots, y(n-1)$. (6p)

c) What is $\hat{z}_3$ if $y(0) = 1, y(1) = 0.5, y(2) = 1, \alpha = \sigma = 2$? (3p)

d) For what values of α, σ does the best estimate $\hat{z}_n$ equal the arithmetic mean $\frac{1}{n}\sum_{t=0}^{n-1} y(t)$? (3p)

Good luck!