KTH Matematik

Solution to Exam October 24, 2008 in SF2832 Mathematical Systems Theory.

Disclaimer: We reserve the right to correct typos in the document.

Allowed books: Anders Lindquist & Janne Sand, An Introduction to Mathematical Systems Theory, class notes and β mathematics handbook.

Solution methods: All conclusions should be carefully motivated.

Note! Your personal number must be stated on the cover sheet. Number your pages and write your name on each sheet that you turn in!

You need 45 points credit (including your bonus) to pass the exam.

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1. Determine if each of the following statements is true or false. You must justify your answers.

(a) Consider $\dot{x} = Ax$ and assume $A^T = -A$. True or False: $x(t)$ always moves in a circle as long as $x(0) \neq 0$. .... (5p)
Answer: true, since $x^T(t)x(t) = x^T(0)x(0)$.

(b) True or False: Given a minimal realization (A, B, C), $(A + BLC, B, C)$ is also minimal with any L. .... (5p)
Answer: true, since controllabity does not change with feedback control and observability does not change with output feedback.

(c) Consider $\dot{x} = Ax$. Assume the equation $Ax = 0$ has more than one solution.
True or False:

(i) The system may be asymptotically stable. .... (3p)
Answer: false.

(ii) There exists an initial condition $x_0 \neq 0$ and a finite $T \geq 0$, such that $e^{At}x_0 = 0,\ \forall t \geq T$. .... (2p)
Answer: false since $det e^{At} \neq 0$.

(d) Consider a controllable and observable system

$$\dot{x} = Ax + bu,$$
$$y = Cx,$$

where $x \in R^n$, $u \in R$ and $y \in R^p$, and $p < n$. True or False: arbitrary pole assignment is never possible with output feedback $u = Ly$. .... (5p)
Answer: true, since we only have at most $p < n$ **parameters to tune the eigenvalues.**

2. Consider :

$$\dot{x} = \begin{pmatrix} -(1+a) & -a \\ 1 & 0 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u$$
$$y = \begin{pmatrix} 0 & 1 \end{pmatrix} x.$$

(a) Is the system minimal? .... (6p)

Answer: yes.

(b) Determine the state transition matrix e^{At} of the system. .... (8p)

Answer: straight forward calculation.

(c) Discuss when the system is BIBO stable. .... (6p)

Answer: Since the system is minimal, we need A to be a stable matrix, which implies $a > 0$.

3. Consider the linearized system of attitude control for a spacecraft:

$$\begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}$$

$$\begin{pmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3 \end{pmatrix} = \sum_{i=1}^{m} b_i u_i,$$

where ϕ, θ, ψ describe the angular position of the spacecraft, and ω_i are angular velocities. Each control u_i is implemented by an opposing pair of gas jets. b_i depends on where the gas jets are placed.

(a) At least how many controls are needed in order to make the system controllable? (6p)

Answer: three.

(b) At least how many measurements are needed in order to make the system observable? .... (6p)

Answer: three.

(c) Design a set of b_i such that the system is controllable and design a state feedback control so that the poles of the closed-loop system are placed at $\{-1, -1, -1, -1, -1, -1\}$. .... (8p)

Answer: we can take $\{b_1, b_2, b_3\} = I_3$. Then the system consists of three subsystems of the following type: $\dot{x}_1 = x_2, \dot{x}_2 = u$. $u = -x_1 - 2x_2$ will assign the eigenvalues to $\{-1, -1\}$.

4. Consider the optimal control problem for a controllable and observable system

$$\min_u J = \int_0^\infty (x^T Q x + u^2) dt \quad \text{s.t.} \quad \dot{x} = Ax + Bu, \; x \in R^n, \; u \in R^m, \; x(0) = x_0.$$

Suppose P^* is the positive definite solution for the associated algebraic Riccati equation.

(a) Show the following linear matrix inequality (LMI) holds:

$$\begin{bmatrix} A^T P^* + P^* A + Q & P^* B \\ B^T P^* & I \end{bmatrix} \geq 0.$$

. . . . (8p)

Answer: we need to show for any $z = (x^T, u^T)^T$, $z^T F(P^*) z \geq 0$. The left hand side is equal to $\|(B^T P^* x + u)\|^2$

(*b*) Define

$$F(P) = \begin{bmatrix} A^T P + PA + Q & PB \\ B^T P & I \end{bmatrix},$$

show for any real symmetric matrix P,

$$\text{rank} F(P) = m + \text{rank}(A^T P + PA + Q - PBB^T P)$$

(which implies that P^* minimizes the rank of $F(P)$.) . . . . (5p)

Proof: Since $F(P)z = 0$ is equivalent to $\begin{bmatrix} A^T P + PA + Q - PBB^T P & 0 \\ B^T P & I \end{bmatrix} z = 0$.

(*c*) Now let $A = a > 0,\ B = 1,\ Q = q$, compute the optimal control $u = kx$. (5p)

Answer: straight forward.

(*d*) What is the closed-loop pole as $q \to 0$? . . . . (2p)

Answer: $-a$.

5. Let z be the outcome of a random variable with distribution $N(0, \alpha^2)$ (i.e., $E\{z\} = 0,\ E\{z^2\} = \alpha^2$). We would like to determine the value of z by a set of noisy measurements

$$y(t) = z + w(t) \text{ for } t = 0, 1, \ldots, n-1$$

where $w(t) \in N(0, \sigma^2)$ are independent of each other and of z.

a) Determine $P(n) = E\{(z - \hat{z}_n)^2\}$ as a function of α, σ, n, where $\hat{z}_n$ is the optimal estimate of z based on measurements up to time instant $n-1$.
(*Hint:* note that $f(k+1)^{-1} = f(k)^{-1} + \delta \Rightarrow f(k) = \frac{1}{f(0)^{-1} + k\delta}$.) (8p)

b) Find $\hat{z}_n$ of z in terms of $y(0), y(1), \ldots, y(n-1)$. (6p)

c) What is $\hat{z}_3$ if $y(0) = 1, y(1) = 0.5, y(2) = 1, \alpha = \sigma = 2$? (3p)

d) For what values of α, σ does the best estimate $\hat{z}_n$ equal the arithmetic mean $\frac{1}{n}\sum_{t=0}^{n-1} y(t)$? (3p)

Solution: One way to solve this is to use the Kalman filter for the system described above with $z = z(t)$ for $t = 0, \ldots$

$$\begin{aligned} z(t+1) &= z(t) \\ y(t) &= z(t) + \sigma v(t), \end{aligned}$$

where v(t) is normalized white noise, and initially $P(0) = E(z(0)^2) = \alpha^2$. Namely, $A = 1, B = 0, C = 1, D = \sigma$. Then the Kalman filter is given by

$$\hat{x}(t+1) = \hat{x}(t) + K(t)(y(t) - \hat{x}(t)) = \hat{x}(t)(1 - K(t)) + K(t)y(t),$$

where $K(t) = \frac{P(t)}{P(t)+\sigma^2}$. The update equation for $P(t)$ is

$$P(t+1) = P(t) - \frac{P(t)^2}{P(t)+\sigma 2} = \frac{1}{P(t)^{-1}+\sigma^{-2}}.$$

Solving the recursion, using that $P(0) = \alpha^2$, we get

$$P(t) = \frac{1}{\alpha^{-2}+t\sigma^{-2}},$$

which equals $P(t) = E\{(\hat{z}_n - z)2\}$, and answers a. The Kalman gain is then

$$K(t) = \frac{1}{1+t+\alpha^{-2}\sigma^2}.$$

We see that

$$\begin{aligned}
\hat{z}(1) &= K(0)y(0) \\
\hat{z}(2) &= (1-K(1))K(0)y(0) + K(1)y(1) = (y(0)+y(1))K(1) \\
\hat{z}(3) &= (1-K(2))K(1)(y(0)+y(1)) + K(2)y(2) = (y(0)+y(1)+y(2))K(2)
\end{aligned}$$

and so on, by noting that $K(n) = (1-K(n))K(n-1)$. I.e.,

$$\hat{z}(n) = K(n-1)\sum_{t=0}^{n-1} y(t).$$

This answers b)). In c) we get by plugging in the values

$$\hat{z}(3) = \frac{1+0.5+1}{4} = 0.625.$$

To answer d), $K(n-1) = \frac{1}{n}$ whenever $\alpha^{-2}\sigma^2 = 0$. I.e., when $\sigma = 0$ or $\alpha = \infty$.