KTH Matematik

Theory Project 3: Balanced Realizations
Mathematical Systems Theory, SF2832
Fall 2007

Satisfactory completion of this project gives three (3) bonus credits for this year's final exam. The examination of the project is *oral and written.* Firstly the results should be presented in a written report (use a word processor) and secondly each student should be prepared to answer question about the report when it is handed back. Hand in the report at the latest **October 10, 5:00pm**. Cooperation in groups of no more than two students is allowed. Only one report for each group is required.

For some problems you will use "Control System Toolbox" in MATLAB. Names written with bold font are command names in MATLAB.
Write **help control** to get a list of available functions in the "Control System Toolbox" or use the help browser.

In this project we will investigate a systematic method for approximating a high order system with a lower order system which has similar controllability and observability properties. The idea is to first find a realization in which the observability and controllability Gramians are the same and diagonal. States corresponding to small diagonal entries in these Gramians are barely controllable and observable and may thus be removed.

Controllability and Observability Gramians

1. Consider a linear time-invariant system with the n-dimensional realization

$$G(s) = C(sI - A)^{-1}B + D \stackrel{\text{def}}{=} \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

where we assume that A is a stable matrix, i.e. $\mathrm{re}\,\lambda_i(A) < 0$, $i = 1, \ldots, n$. The controllability and observability Gramians are defined as

$$W = \int_0^\infty e^{At}BB^Te^{A^Tt}dt,$$
$$M = \int_0^\infty e^{A^Tt}C^TCe^{At}dt,$$

and can be computed as the solutions to the Lyapunov equations

$$AW + WA^T + BB^T = 0$$
$$A^TM + MA + C^TC = 0$$

Let us consider an equivalent realization (where T is a nonsingular matrix)

$$\left[\begin{array}{c|c} \bar{A} & \bar{B} \\ \hline \bar{C} & \bar{D} \end{array}\right] = \left[\begin{array}{c|c} TAT^{-1} & TB \\ \hline CT^{-1} & D \end{array}\right] \tag{1}$$

and its associated controllability and observability Gramians $\bar{W}$ and $\bar{M}$.

Solve the following problem:

(a) Show that $\bar{W}\bar{M} = T\bar{W}\bar{M}T^{-1}$.

Balanced Realizations

2. We will next consider a particular similarity transformation T that results in controllability and observability Gramians that are equal and diagonal.

From now on we make the assumption that the system is observable, i.e. W and M are positive definite. This implies that the controllability matrix can be factorized as $W = R^T R$, where R is a nonsingular matrix. Then $R^T M R$ is positive definite and symmetric and hence it has a spectral decomposition (eigenvalue decomposition) on the form[1]

$$RMR^T = U\Sigma^2 U^T$$

where U is an orthogonal matrix, i.e. $U^T U = I$, and $\Sigma = \text{diag}(\sigma_1, \dots, \sigma_n)$ with $\sigma_1 \geq \sigma_2 \geq \dots \sigma_n > 0$. The σ_k are often called *singular values*.

Solve the following problem:

(b) Let $T = \Sigma^{1/2} U^T (R^T)^{-1}$. Show that the realization in (1) has controllability and observability Gramians $\bar{W} = \bar{M} = \Sigma$

Balanced Truncation

3. Suppose we have done the similarity transformation in (b) and the singular values satisfy $\Sigma = \text{diag}(\Sigma_1, \Sigma_2)$, where

$$\Sigma_1 = \text{diag}(\sigma_{11}, \dots, \sigma_{1n_1})$$
$$\Sigma_2 = \text{diag}(\sigma_{21}, \dots, \sigma_{2n_2})$$

where $n_1 + n_2 = n$ and

$$\sigma_{11} \geq \sigma_{12} \geq \dots \geq \sigma_{1n_1} >> \sigma_{21} \geq \sigma_{22} \geq \dots \geq \sigma_{2n_2} \tag{2}$$

Let the corresponding balanced realization have the block partitioning

$$\left[\begin{array}{c|c} \bar{A} & \bar{B} \\ \hline \bar{C} & \bar{D} \end{array}\right] = \left[\begin{array}{cc|c} \bar{A}_{11} & \bar{A}_{12} & \bar{B}_1 \\ \bar{A}_{21} & \bar{A}_{22} & \bar{B}_2 \\ \hline \bar{C}_{11} & \bar{C}_{12} & \bar{D} \end{array}\right] \tag{3}$$

and the corresponding state vector $\bar{x} = \begin{bmatrix} \bar{x}_1^T & \bar{x}_2^T \end{bmatrix}^T$. The states $\bar{x}_2$ are poorly controllable and poorly observable compared to the states in $\bar{x}_1$ due to (2). It is thus reasonable to believe that the states $\bar{x}_2$ can be removed without changing the input-output map significantly. You will next derive the corresponding reduced order realization

(c) Let $\bar{x}_2 = 0$. What is the reduced order realization?

(d) Let $\dot{\bar{x}}_2 = 0$. What is the reduced order realization?

[1] This type of eigenvalue decomposition is also called singular value decomposition (SVD). The SVD can also be defined for non-symmetric and non-square matrices.

In order to justify the reduced order realization we consider a realization on the form (3) with $\bar{W} = \bar{M} = \text{diag}(\Sigma_1, 0)$, i.e. the singular values corresponding to the states $\bar{x}_2$ are zero.

(e) Show that $G_r(s) = \left[\begin{array}{c|c} \bar{A}_{11} & \bar{B}_1 \\ \hline \bar{C}_1 & \bar{D} \end{array}\right]$ is a realization, i.e. $G_r(s) = G(s)$, where $G(s) = \left[\begin{array}{c|c} \bar{A} & \bar{B} \\ \hline \bar{C} & \bar{D} \end{array}\right]$.

When $\Sigma_2 > 0$ but the bound in (2) holds then it is possible to prove that

$$\sup_{\omega \in \mathbf{R}} |G(j\omega) - G_r(j\omega)| \leq 2(\sigma_{21} + \ldots \sigma_{2n_2})$$

which when all these singular values are small implies that we have good approximation across all frequencies in the Bode diagram.

Applications

4. Consider the system

$$G(s) = \frac{458}{(s+1)(s^2 + 30s + 229)}$$

You should now use Matlab commands for model reduction of this system.

(f) Use the Matlab commands `balreal` and `modred` to derive second and first order approximations of this system. Compare the full order system with the reduced order systems by ploting all systems in a Bode diagram. The Bode plot and the obtained reduced order models should be included in your report.

Good Luck! Don't hesitate to ask (by email or phone) if anything is unclear.