KTH Matematik

Theory Project 1: Linear Matrix Equations
Mathematical Systems Theory, SF2832
Fall 2007

Satisfactory completion of this project gives three (3) bonus credits for this year's final exam. The examination of the project is *oral and written*. Firstly the results should be presented in a written report (use a word processor) and secondly each student should be prepared to answer question about the report when it is handed back. Hand in the report at the latest **October 10, 5:00pm**. Cooperation in groups of no more than two students is allowed. Only one report for each group is required.

For some problems you will use "Control System Toolbox" in MATLAB. Names written with bold font are command names in MATLAB.
Write **help control** to get a list of available functions in the "Control System Toolbox" or use the help browser.

The Sylvester Equation

1. The Sylvester equation is a linear matrix equation which is used in stability analysis, feedback design, observer design, and model reduction. We consider the following version

$$FX + XG + H = 0 \tag{1}$$

where $F, G, H \in \mathbf{R}^{n\times n}$ are given matrices and $X \in \mathbf{R}^{n\times n}$ is the variable to be determined. In order to understand when this equation has a unique solution and how it can be computed we will rewrite it in terms of the Kronecker product and the Vec operator. The Kronecker product between two $n \times n$ matrices A and B is defined as the $n^2 \times n^2$ matrix

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \dots & a_{1n}B \\ \vdots & \vdots & & \vdots \\ a_{n1}B & a_{n2}B & \dots & a_{nn}B \end{bmatrix}$$

If matrix $C \in \mathbf{R}^{n\times n}$ has columns c_i, i.e., $C = \begin{bmatrix} c_1 & \dots & c_n \end{bmatrix}$ then $\mathrm{Vec}(C)$ is the n^2 dimensional vector

$$\mathrm{Vec}(C) = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

We will use that if A has eigenvalues $\lambda_1, \dots, \lambda_n$ and B has eigenvalues $\mu_1, \dots, \mu_n$ then the Kronecker sum (where I is an $n \times n$ identity matrix)

$$I \otimes A + B \otimes I$$

has eigenvalues $\lambda_i + \mu_j$, $i = 1, \dots, n$, $j = 1, \dots, n$.

Solve the following problems

(a) Show that (1) can be written as

$$(I \otimes F + G^T \otimes I)\mathrm{Vec}(X) + \mathrm{Vec}(H) = 0.$$

Hint: Consider an arbitrary column of (1). By combining the columns you get the above expression. The Sylvester equation is thus equivalent to standard linear equation.

(b) Show that the Sylvester equation has a unique solution if and only if $\lambda_i + \mu_j \neq 0$, $i = 1, \dots, n$, $j = 1, \dots, n$, where λ_i, μ_j are eigenvalues of F and G, respectively.

(c) The Lyapunov equation $A^T P + PA + Q = 0$ is a special case of the Sylvester equation. Use (b) to show that there exists a unique solution if A is a stable matrix (all eigenvalues in the open left half plane).

Matrix Differential Equations

2. We will next study a linear matrix differential equation on the form

$$\dot{X}(t) = FX(t) + X(t)G + H, \qquad X(t_0) = X_0 \tag{2}$$

We will investigate the closed solution formula and learn how the reachability Grammian can be obtained as the solution to such matrix differential equations.

(d) Verify that the solution of (2) satisfies the following matrix variation of constants formula:

$$X(t) = e^{F(t-t_0)} X_0 e^{G(t-t_0)} + \int_{t_0}^{t} e^{F(t-\tau)} H e^{G(t-\tau)} d\tau. \tag{3}$$

Remark: It is also possble to obtain the coefficients of $X(t)$ from the standard vector differential equation

$$\frac{d\mathrm{Vec}(X(t))}{dt} = (I \otimes F + G^T \otimes I)\mathrm{Vec}(X(t)) + \mathrm{Vec}(H)$$

The solutions are identical, which can be proven using Kronecker product formulas.

(e) Derive a matrix differential equation on the form (2) for the reachability Grammian $W(t_0, t) = \int_{t_0}^{t} e^{A(t-\tau)} BB^T e^{A^T(t-\tau)} d\tau$.

Stationary Solutions

3. In the case when F and G are stable matrices the expression in (3) converges as $t \to \infty$. The limit solution

$$X = \int_0^{\infty} e^{Ft} H e^{Gt} dt$$

is called stationary solution.

(f) Show that the stationary solution is a solution of the Sylvester equation in (1).

(*g*) Derive a Lyapunov equation for computing the following stationary reachability Grammian $W = \int_0^\infty e^{At}BB^Te^{A^Tt}dt$.

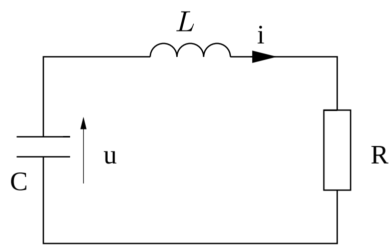
Applications

4. We next apply the above theory to two examples.

(*h*) Consider the electric circuit in Figure 1. Compute the following energy dissipation in the resistor

$$E = \int_0^\infty i(t)^2 dt$$

as a function of the initial conditions $i(0) = i_0$ and $u(0) = u_0$. The capacitance, the resistance, and the inductance are all set to 1.



Figur 1: Electric circuit

(*i*) [Optional] We have not discussed why the general form of the Sylvester equation is useful. One possible application is state feedback design. Let

$$\dot{x} = Ax + Bu$$

be a given system where $A \in \mathbf{R}^{n\times n}$ and $B \in \mathbf{R}^{n\times m}$. We want to find a state feedback control $u = Kx$ such that $A + BK$ has eigenvalues in some desired locations. The following is an algorithm to design K:

(*i*) Choose matrix A_{cl} with eigenvalues in desired locations.

(*ii*) Pick some matrix $\bar{K}$ such that the Sylvester equation

$$AX - XA_{cl} = -B\bar{K}$$

has a nonsingular solution X, i.e X must be invertible.

(*iii*) Then $K = \bar{K}X^{-1}$ is a state feedback with the desired properties.

Show that $A + BK$ has the same closed loop eigenvalues as A_{cl}.

In the course Geometrisk styrteori, SF2842, you will learn more more about the application of Sylvester equations for the solution of various nonlinear and nonlinear control and estimation problems.

Good Luck! Don't hesitate to ask (by email or phone) if anything is unclear.