KTH Matematik

Theory Project 2: System Interconnection
Mathematical Systems Theory, SF2832
Fall 2007

Satisfactory completion of this project gives three (3) bonus credits for this year's final exam. The examination of the project is *oral and written*. Firstly the results should be presented in a written report (use a word processor) and secondly each student should be prepared to answer question about the report when it is handed back. Hand in the report at the latest **October 10, 5:00pm**. Cooperation in groups of no more than two students is allowed. Only one report for each group is required.

For some problems you will use "Control System Toolbox" in MATLAB. Names written with bold font are command names in MATLAB.
Write **help control** to get a list of available functions in the "Control System Toolbox" or use the help browser.

Low order realizations are for computational efficiency reasons important when implementing systems in software. In particular, for a specific system a minimal realization is desired, i.e. a realization that is both reachable and observable. When interconnecting minimal systems, the interconnected system is not always minimal. This is what we will study in this project.

System Interconnection

1. The system

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t)\\ y(t) &= Cx(t) + Du(t)\end{aligned}$$

has the transfer function $G(s) = C(Is - A)^{-1}B + D$ which is sometimes denoted by

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) := C(Is - A)^{-1}B + D.$$

In this part we are going to study interconnections between such systems.

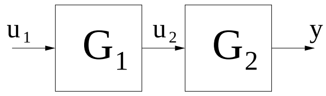
Serial connection

First we will study the serial connection given in Figure 1. With a slight abuse of notation we shall call the two systems G_1 and G_2. The system G_1 is given by

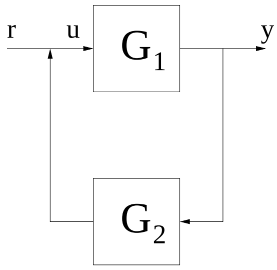
$$\dot{x}_1(t) = A_1x_1(t) + B_1u_1(t) \tag{1}$$
$$u_2(t) = C_1x_1(t) + D_1u_1(t) \tag{2}$$

and the system G_2 is given by

$$\dot{x}_2(t) = A_2x_2(t) + B_2u_2(t) \tag{3}$$
$$y(t) = C_2x_2(t) + D_2u_2(t). \tag{4}$$



Figur 1: Serial interconnected system



Figur 2: Feedback system

The transfer functions of the systems are

$$G_i(s) = \left(\begin{array}{c|c} A_i & B_i \\ \hline C_i & D_i \end{array}\right) \text{ for } i = 1, 2.$$

By substituting (2) into (3) and (4) we get

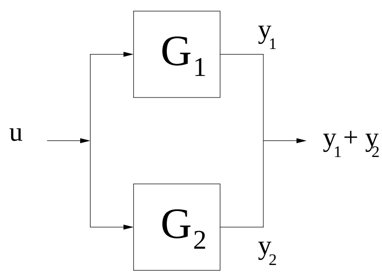
$$\begin{aligned} \dot{x}_1(t) &= A_1 x_1(t) + B_1 u_1(t) \\ \dot{x}_2(t) &= B_2 C_1 x_1(t) + A_2 x_2(t) + B_2 D_1 u_1(t) \\ y(t) &= D_2 C_1 x_1(t) + C_2 x_2(t) + D_2 D_1 u_1(t). \end{aligned}$$

and hence the transfer function of the interconnected system is

$$\left(\begin{array}{cc|c} A_1 & 0 & B_1 \\ B_2 C_1 & A_2 & B_2 D_1 \\ \hline D_2 C_1 & C_2 & D_2 D_1 \end{array}\right).$$

For the feedback system, given in Figure 2, let the system G_1 be given by

$$\begin{aligned} \dot{x}_1(t) &= A_1 x_1(t) + B_1 u(t) \\ y(t) &= C_1 x_1(t), \end{aligned} \tag{5}$$



Figur 3: Parallel interconnected system

and let the system G_2 be given by

$$\begin{aligned}\dot{x}_2(t) &= A_2x_2(t) + B_2y(t)\\ v(t) &= C_2x_2(t) + D_2y(t).\end{aligned}$$

From Figure 2 $u(t) = r(t) + v(t)$. For simplicity we have assumed that $D_1 = 0$.

(a) Find the transfer function from $r(t)$ to $y(t)$?

(b) Find the transfer function for the parallel system in figure 3, when

$$G_i(s) = \left(\begin{array}{c|c} A_i & B_i \\ \hline C_i & D_i \end{array}\right) \quad \text{for } i = 1, 2?$$

Reachability

2. A system

$$\dot{x}(t) = Ax(t) + Bu(t)$$

is reachable if for any given time t_0, initial condition $x(t_0) = x_0$ and final condition x_1, it is possible to find a bounded control $u(t)$ and a time t_1 such that $x(t_1) = x_1$.

This is an important concept and there are many different ways to characterize this.

The following are equivalent:

(i) The pair (A, B) is reachable,

(ii) The matrix $[B\ AB\ \dots\ A^{n-1}B]$ is full rank,

(iii) The reachability gramian

$$W(t_0, t_1) = \int_{t_0}^{t_1} e^{At}BB^Te^{A^Tt}dt$$

is strictly positive definite for all $t_0 < t_1$,

(iv) If $v^TA = \lambda v^T$ and $v \neq 0$, then $v^TB \neq 0$.

(v) The matrix $[\lambda I - A\ B]$ is full rank for all $\lambda \in \mathbb{C}$.

In the course we have shown that the properties $(i), (ii)$ and (iii) are equivalent. Here we will prove that (iv) and (v) are equivalent with (ii).

(a) First we will show that $(ii) \Rightarrow (iv)$. For this, we use a proof by contradiction. Assume that (iv) is not true, i.e. that there exist a vector $v \neq 0$ such that $v^TA = \lambda v^T$ and $v^TB = 0$. Show that then $v^T[B\ AB\ \dots\ A^{n-1}B] = 0$ and hence (ii) does not hold.

(b) To show that $(iv) \Rightarrow (ii)$, we again use proof by contradiction. Assume that the system is not reachable and put it into the form anologous to the kalman decomposition, i.e.

$$A = \begin{pmatrix} A_c & A_{c\bar{c}} \\ 0 & A_{\bar{c}} \end{pmatrix}, \quad B = \begin{pmatrix} B_c \\ 0 \end{pmatrix}.$$

Show that $(iv) \Rightarrow (ii)$.

(c) Show that (iv) and (v) are equivalent.

Feel free to use an other line of proof if you find it easier, as long as you only refer to results proved in the course.

Reachability and Minimality of Interconnected Systems

3. (a) Show that the systems

$$\begin{aligned} \dot{x}_1(t) &= -2x_1(t) + u_1(t) \\ u_2(t) &= -x_1(t) + u_1(t) \end{aligned}$$

and

$$\begin{aligned} \dot{x}_2(t) &= -3x_2(t) - u_2(t) \\ y(t) &= x_2(t) + u_2(t) \end{aligned}$$

are minimal, but the serial connection of them is not.

(b) Find two minimal systems G_1 and G_2 such that the state space representation of the feedback system in Figure 2 is not minimal.

(c) Show that if the system G_1 given by (5) is minimal. Then the feedback system with state feedback (i.e. $A_2 = 0$,$B_2 = 0$, and $C_2 = 0$) is minimal.

(d) Is a parallel connection of two minimal systems always minimal?

Matlab part

4. The mat-file `P_PHB.mat` contains the two systems G1 and G2, see `http://www.math.kth.se/optsyst/grundutbildning/kurser/SF2832/kursPM.html`.

Consider serial connection, feedback connection, and parallel connection. Use matlab to do the following:

(a) Find out if the interconnection is minimal.

(b) If it is not, find a minimal state space realization of it.

A matlab command that might be handy is “minreal”. Note that multiplication and addition is defined for systems.

From these four excercises, it is clear that in some cases the interconnected systems give a minimial realization, but in other cases it does not. When one has obtained a nonminimal system one has to be carefull and consider why it is not minimal. Is it due to lack of reachability or lack of observability? Are the unreachable/unobservable modes stable or unstable? Do the states represent actual physical properties of the system? This will be considered later in the course.
Good Luck! Don’t hesitate to ask (by email or phone) if anything is unclear.