KTH Matematik

Solution to Exam in SF2832 Mathematical Systems Theory, March 2013.

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Allowed material: Anders Lindquist & Janne Sand, An Introduction to Mathematical Systems Theory, your own class notes, and β mathematics handbook.

Solution methods: All conclusions should be carefully motivated.

Note! Your personal number must be stated on the cover sheet. Number your pages and write your name on each sheet that you turn in!

You need 45 points credit (including your still valid bonus) to pass the exam. The other grade limits are listed on the course home page.

Read this before you start: 1. The problems are NOT necessarily ordered in terms of difficulty. 2. Problem 4(c) requires some calculation no matter which method you take.

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1. Determine if each of the following statements is true or false. You must justify your answers. **All matrices involved are assumed to be constant matrices unless otherwise specified**.

(a) Consider an n-dimensional time-varying system $\dot{x} = A(t)x$, where $A(t)$ is continuous. Then $\|x(t)\|^2 = \|x(t_0)\|^2 \ \forall t \geq t_0$ as long as $A^T(t) = -A(t) \ \forall t \in R$. (5p)

Answer: True, since $\frac{d}{dt}\|x(t)\|^2 = 0$.

(b) Consider an unobservable system $\dot{x} = Ax, y = Cx$, where $x \in R^n$. Given two distinct points $x_i \notin ker\ \Omega$ (the unobservable subspace) $i = 1, 2$, then $Ce^{At}x_1 - Ce^{At}x_2$ can not be identically zero for all $t > 0$. .... (5p)

Answer: False. We can let $x_2 = x_1 + x_0$, where $x_0 \in ker\ \Omega$.

(c) Consider $\dot{x} = Ax + bu$, where $x \in R^n, u \in R$. A necessary condition for arbitrary assignment of eigenvalues of $A + bk$ is that rank $A \geq n - 1$. .... (5p)

Answer: True. Arbitrary pole assignment is possible iff (A, b) is controllable. If (A, b) is controllable, we can rewrite the system after linear transformation into the standard controllable form where $\bar{A}$ has rank at least $n - 1$.

(d) Consider the optimal control problem for $\dot{x} = Ax + Bu$: $\min_u \ x^T(t_1)Sx(t_1) + \int_{t_0}^{t_1}(x^T Qx + u^T Ru)dt$, where $S \geq 0$, $Q \geq 0$ and $R > 0$. Then $P(t)$ is positive definite for $t_0 \leq t < t_1$ if and only if S is positive definite, where $P(t)$ is the solution to the corresponding dynamical Riccati equation. ....(5p)

Answer: False. For example, $P(t)$ is positive definite on $[t_0, t_1)$ even for $S = 0$ if $Q > 0$ (Let $A = 0$, $B = 0$ for example).

2. Consider :

$$\begin{aligned} \dot{x} &= Ax + bu \\ y &= cx, \end{aligned}$$

where

$$A = \begin{bmatrix} 0 & a_1 \\ a_2 & 0 \end{bmatrix}, \ b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ c = \begin{bmatrix} 1 & 1 \end{bmatrix}, a_1 \neq a_2, a_1 a_2 > 0$$

(*a*) Find the state transition matrix e^{At}. .... (5p)
Answer: $e^{At} = \begin{pmatrix} e_1(t) & ke_2(t) \\ k^{-1}e_2(t) & e_(t) \end{pmatrix}$, where $e_1(t) = \frac{1}{2}(exp(\sqrt{a_1a_2}t)+exp(-\sqrt{a_1a_2}t))$, $e_2(t) = \frac{1}{2}(exp(\sqrt{a_1a_2}t) - exp(-\sqrt{a_1a_2}t))$, $k = \frac{\sqrt{a_1}}{\sqrt{a_2}}$.

(*b*) Design a feedback control $u = kx$ such that the closed-loop poles are $\{-1, -2\}$. (5p)
Answer: $k_1 = -3 - k_2,\ k_2 = \frac{2-3a_1+a_1a_2}{a_1-a_2}$.

(*c*) Find an open-loop control $u(t)$ (i.e. $u(t)$ should be a function of time and the initial states) in any way you can, such that whenever $cx(0) = 0$, $y(t) = 0$ for all $t \geq 0$. .... (6p)
Answer: $u = \frac{a_1-a_2}{2} exp(-\frac{1}{2}(a_1 + a_2)t)x_0$, where $x_1(0) = -x_2(0) = x_0$.

(*d*) What happens to $x(t)$ as $t \to \infty$ when the control designed in (c) is applied? (4p)
Answer: $x(t) \to 0$ if $a_1 + a_2 > 0$.

3. Consider the transfer matrix

$$R(s) = \begin{bmatrix} \frac{\gamma}{s} & \frac{\beta}{s+1} \\ \frac{1}{s} & \frac{1}{s+1} \end{bmatrix},$$

where β, γ are nonzero constants.

(*a*) Find the standard reachable realization. .... (7p)
Answer: omitted.

(*b*) Can the realization in (a) also be observable and why? .... (6p)
Answer: No, since the McMillan degree is 2.

(*c*) For the case $\beta = \gamma$, find a minimal realization of $R(s)$. .... (7p)
Answer: When $\beta = \gamma$, $y_1 = \gamma y_2$. So we just need to find a minimal realization for $r_2(s) = [\frac{1}{s}\ \frac{1}{s+1}]$. The standard observable realization gives

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} u \\ y_2 &= \begin{bmatrix} 1 & 0 \end{bmatrix} x. \end{aligned}$$

One can easily see that it is also controllable. Adding in $y_1 = [\gamma\ 0]x$ we have the minimal realization.

4. Consider the optimal control problem

$$\min_u \int_0^\infty (y^2 + u^2)dt$$

subject to

$$\begin{aligned}\dot{x} &= Ax + bu\\ y &= cx\end{aligned}$$

where,

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad c = \begin{bmatrix} 1 & 1 \end{bmatrix}.$$

(*a*) Show that the optimal control exists. ....(5p)
Proof: the optimal control exists since the system is both controllable and observable.

(*b*) Some times the algebraic Riccati equation (ARE) is easier to solve if we do a coordinate change first. Show that if we let $\bar{x} = Sx$, then in the new coordinate system, $\bar{P} = S^{-T}PS^{-1}$...(5p)
Proof: Replacing A by $S^{-1}\bar{A}S$, b by $S^{-1}\bar{b}$ and c by $\bar{c}S$ in the ARE, we obtain the conclusion.

(*c*) Find a coordinate change and solve the ARE associated with the above optimal control problem in the new coordinates, i.e., find $\bar{P}$ (Hint: Completion of square might be useful for solving the ARE)....................................(10p)
Answer:
Let $\bar{x}_1 = x_1 + x_2,\ \bar{x}_2 = x_2$, and $\bar{P} = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}$, we have

$$-2p_1 - p_2^2 + 1 = 0,\ 2p_1 - 2p_2 - 2p_2p_3 = 0,\ 2p_2 - p_3^2 = 0.$$

Using eqns (1) and (3), the second equation becomes $p_3^2 + p_2^2 + 2p_2p_3 - 1 = 0$, i.e. $(p_2 + p_3)^2 = 1$, which gives $p_2 = \pm 1 - p_3$. Since $p_3 > 0$, using eqn. (3) we see that $p_2 = 1 - p_3$. Then, $p_3 = -1 + \sqrt{3},\ p_2 = 2 - \sqrt{3},\ p_1 = 2\sqrt{3} - 3$.

5. (*a*) Consider a controllable and observable SISO system:

$$\begin{aligned}\dot{x} &= Ax + bu\\ y &= cx.\end{aligned}$$

Let $g(s) = c(sI - A)^{-1}b = \frac{s^q + p_1 s^{q-1} + \cdots + p_q}{s^n + d_1 s^{n-1} + \cdots + d_n}$ be the transfer function. Now we want to use feedback control $u = kx$ to make the system unobservable "as much as possible". Let Ω_k denote the observability matrix for $(c, A + bk)$. Show

$$\max_k \ dim\ ker(\Omega_k) = q,$$

where "dim" denotes dimension and "ker" denotes kernel. ....(10p)

Answer: It is zero/pole cancellation that induces unobservability. Since the system is controllable, we can find $u = k^*x$ to assign poles such that all zeros are canceled. Now let $u = k^*x + v$, then

$$\begin{aligned} \dot{x} &= (A + bk^*)x + bv \\ y &= cx. \end{aligned} \tag{1}$$

A minimal realization of (1) clearly has dimension $n - q$ (the transfer function after zero/pole cancellation). By Kalman decomposition we see that $ker\ \Omega$ for (1) has dimension q.

(*b*) Let x be the outcome of a random variable with distribution $N(0, \alpha^2)$ (i.e., $E\{x\} = 0,\ E\{x^2\} = \alpha^2$). We would like to estimate the value of x by a set of noisy measurements

$$y(t) = x + w(t) \text{ for } t = 0, 1, \ldots, n-1$$

where $w(t) \in N(0, \sigma^2(t))$ are independent of each other and of x. Determine $P(n) = E\{(x - \hat{x}_n)^2\}$ as a function of $\alpha, \sigma(1), \cdots, \sigma(n-1)$, where $\hat{x}_n$ is the optimal estimate of x based on measurements up to time instant $n - 1$. (Hint: one could normalize the noise). .... (10p)

Answer: We let $w(t) = \sigma(t)\tilde{w}(t)$, where $w(t) \in N(0, 1)$, then $y(t) = x + \sigma(t)\tilde{w}(t)$. Thus $P(t+1) = P(t) - \frac{P(t)}{P(t)+\sigma(t)^2}$, or $P(t+1)^{-1} = P(t)^{-1} + \sigma(t)^{-2}$. Thus $P(n)^{-1} = \sum_{i=0}^{n-1} \sigma(i)^{-2} + \alpha^{-2}$.