KTH Matematik

Solution to Exam of SF2832 Mathematical Systems Theory, 16 January 2014.

Examiner: Xiaoming Hu, tel. 790 7180.

Allowed material: Anders Lindquist & Janne Sand, An Introduction to Mathematical Systems Theory, your own class notes, and β mathematics handbook.

Solution methods: All conclusions should be carefully motivated.

Note! Your personal number must be stated on the cover sheet. Number your pages and write your name on each sheet that you turn in!

You need 45 points credit (including your still valid bonus) to pass the exam. The other grade limits are listed on the course home page.

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1. Determine if each of the following statements is true or false. You must justify your answers. **All matrices involved are assumed to be constant matrices unless otherwise specified**.

(*a*) Consider an n-dimensional time-varying system $\dot{x} = A(t)x, x(t_0) = x_0$, with $A(t)$ being continuous on $(-\infty, \infty)$. Let $\dot{z} = -A(t)^T z, z(t_1) = z_1$. Then $z^T(t)x(t) = z_1^T \Phi(t_1, t_0)x_0, \ \forall t \in [t_0, t_1]$. . . . . (5p)
Answer: True. Since $\frac{d}{ds}\Phi^T(t,s) = (-\Phi(t,s)A(s))^T = -A^T\Phi^T$, $\Phi_z(t,s) = \Phi^T(s,t)$. Thus $z(t) = \Phi_z(t,t_1)z_1 = \Phi^T(t_1,t)z_1$ and the conclusion follows.

(*b*) Consider $\dot{x} = Ax + bu$, where $x \in R^n, u \in R$. (A, b) is controllable only if (necessary condition) there exists a nonsingular matrix T such that

$$TAT^{-1} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ a_1 & a_2 & a_3 & \cdots & a_n \end{pmatrix},$$

where $a_1, \cdots, a_n$ are real coefficients. . . . .(5p)
Answer: True, since if (A, b) is controllable we can convert the system into the standard controllable form.

(*c*) Consider $\dot{x} = Ax + bu, y = cx$, where $x \in R^n, u \in R, y \in R$ and $c \neq 0$. If (A, b) is controllable, then there exists $p \in R^n$, such that $(c, A + bp^T)$ is observable. (5p)
Answer: True, since we can always find $u = p^T x + v$ such that the closed-loop poles do not cancel the zeros.

(*d*) Consider the Riccati differential equation:

$$\begin{aligned} \dot{P}(t) &= -A^T P(t) - P(t)A + P(t)BR^{-1}B^T P(t) - C^T C \\ P(t_1) &= P_1, \end{aligned}$$

where R is positive definite. If P_1 is only positive semidefinite ($det\ P_1 = 0$), then $P(t)$ is not positive definite either for $t < t_1$. . . . . (5p)
Answer: Obviously false.

2. Consider a two dimensional SISO system:

$$\begin{aligned} \dot{x} &= Ax + bu \\ y &= cx, \end{aligned}$$

namely $x \in R^2, u \in R, y \in R$. Assume A is given and $A^T = -A \neq 0$.

(a) Find the state transition matrix e^{At} (find expression for each element in e^{At}). (5p)
Answer: The fact $A^T = -A$ implies $A = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}$ and the rest is omitted.

(b) Find conditions on b and c such that (A, b, c) is a minimal realization. . . . (5p)
Answer: $b \neq 0, c \neq 0$.

(c) Show $A - bb^T$ is a stable matrix if and only if (A, b) is controllable. . . . . . (5p)
Answer: Let $P = \frac{1}{2}I$, then $(A - bb^T)P + P(A - bb^T)^T + bb^T = 0$. Thus it is a stable matrix if (A, b) controllable. (A, b) is not controllable if $b = 0$, then A itself is obviously not a stable matrix.

(d) Assume (A, b, c) is a minimal realization. Show that the system can be asymptotically stabilized by output feedback $u = ky$ if $cb \neq 0$ and $cb \cdot cAb \geq 0$. (5p)
Answer: Let $\bar{x}_1 = cx, \bar{x}_2 = cAx$, and $u = k\bar{x}_1$, then

$$\begin{aligned} \dot{\bar{x}}_1 &= \bar{x}_2 + cbk\bar{x}_1 \\ \dot{\bar{x}}_2 &= -a^2\bar{x}_1 + cAbk\bar{x}_1, \end{aligned}$$

and the conclusion follows.

3. Consider the transfer matrix

$$R(s) = \begin{bmatrix} \frac{\gamma}{s} & \frac{s+2}{s(s+1)} \\ \frac{1}{s} & \frac{s+2}{s(s+1)} \end{bmatrix},$$

where γ is a nonzero constant.

(a) Find the standard reachable realization. . . . . (7p)
Answer: Omitted.

(b) Compute the McMillan degree of $R(s)$. . . . .(6p)
Answer: $\delta R = 3$ if $\gamma \neq 1$, otherwise 2.

(c) For the case $\gamma = 1$, find a minimal realization of $R(s)$. . . . . (7p)
Answer: In this case $\delta R = 2$ and the nontrivial part is to find a minimal realization for $\begin{bmatrix} \frac{1}{s} & \frac{s+2}{s(s+1)} \end{bmatrix}$. The observable realization will do it.

4. Consider the optimal control problem

$$\min_u \int_0^\infty (y^2 + \epsilon^2 u^2) dt$$

subject to

$$\dot{x} = Ax + bu$$
$$y = cx$$

where,

$$A = \begin{bmatrix} a & 1 \\ 1 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad c = \begin{bmatrix} 0 & 1 \end{bmatrix}, \text{ and } a \text{ is constant.}$$

(*a*) Show that the optimal control exists for any $\epsilon > 0$. .... (5p)

Answer: Since the system is both controllable and observable.

(*b*) The "cheap" control problem studies when the optimal control problem is well posed as $\epsilon \to 0$. For the case $a = 0$, show, by solving the algebraic Riccati equation (ARE), that $\lim_{\epsilon\to 0}(A - \epsilon^{-2}bb^T P)$ has one eigenvalue on the imaginary axis, where P is the solution to ARE associated with the optimal control, thus the cheap control is not well posed. .... (7p)

Answer: Plug in $P = \begin{bmatrix} \epsilon\sqrt{1+4\epsilon^2} & 2\epsilon^2 \\ 2\epsilon^2 & \epsilon\sqrt{1+4\epsilon^2} \end{bmatrix}$, then the conclusion follows.

(*c*) For the case $a \neq 0$, show $\lim_{\epsilon\to 0}(A - \epsilon^{-2}bb^T P)$ has eigenvalues at $-|a|, -\infty$, thus the cheap control is well posed. .... (8p)

Answer: For sufficiently small ϵ, we have $P \approx \begin{bmatrix} p_1 & \epsilon p_1 \\ \epsilon p_1 & \epsilon \end{bmatrix}$, where $p_1 = 2a$ if $a > 0$, otherwise 0. The conclusion then follows.

5. (*a*) Consider N linear systems (agents) of dimension n:

$$\dot{x}_i = Ax_i + Bu_i, \ i = 1, \cdots, N,$$

where each agent i is only allowed to use information of type $x_i - x_j$ for control design, where agent j is a "neighbor" of agent i. We say the multi-agent system reaches consensus if under some feedback control we have $x_1(t) = x_2(t) = \cdots = x_N(t)$ as $t \to \infty$. It is shown in the literature that under the assumption of "connected neighborhood graph", the consensus problem is solvable if there exists a feedback matrix K such that $A + \lambda_i BK$ is a stable matrix for each λ_i where $0 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_N$ (eigenvalues of the so-called graph Laplacian). Show there exists indeed such a matrix K if (A, B) is controllable, by using the following algebraic Riccati equation:

$$A^T P + PA - \sigma PBB^T P + Q = 0,$$

with some proper choice of σ (a scalar) and Q. .... (12p)

Answer: Let $\sigma = \lambda_2$ and Q be positive definite, then the ARE has a positive definite solution P. Let $K = -B^T P$, then $(A - \lambda_i BB^T P)^T P + P(A - \lambda_i BB^T P) = -Q - (2\lambda_i - \lambda_2)PBB^T P$. The right hand side is clearly negative definite, thus $A - \lambda_i BB^T P$ is a stable matrix for each i.

(*b*) Let x be the outcome of a random variable with distribution $N(0, \alpha^2)$ (i.e., $E\{x\} = 0,\ E\{x^2\} = \alpha^2$). We would like to estimate the value of x by a set of noisy measurements

$$y(t) = x + w(t) \text{ for } t = 0, 1, \ldots, n-1$$

where $w(t) \in N(0, \sigma^2(t))$ are independent of each other and of x. Determine $P(n) = E\{(x - \hat{x}_n)^2\}$ as a function of $\alpha, \sigma(1), \cdots, \sigma(n-1)$, where $\hat{x}_n$ is the optimal estimate of x based on measurements up to time instant $n-1$. . (8p)

Answer: We let $w(t) = \sigma(t)\tilde{w}(t)$, where $\tilde{w}(t) \in N(0,1)$, then $y(t) = x + \sigma(t)\tilde{w}(t)$. Thus $P(t+1) = P(t) - \frac{P(t)}{P(t)+\sigma(t)^2}$, or $P(t+1)^{-1} = P(t)^{-1} + \sigma(t)^{-2}$. Thus $P(n)^{-1} = \sum_{i=0}^{n-1} \sigma(i)^{-2} + \alpha^{-2}$.