KTH Matematik

Solution to Exam March 15, 2016, SF2832 Mathematical Systems Theory.

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Allowed material: Anders Lindquist & Janne Sand, An Introduction to Mathematical Systems Theory, Per Enqvist, Exercises in Mathematical Systems Theory, your own class notes, and β mathematics handbook.

Solution methods: All conclusions should be carefully motivated.

Note! Your personal number must be stated on the cover sheet. Number your pages and write your name on each sheet that you turn in!

You need 45 points credit (including your still valid bonus) to pass the exam. The other grade limits are listed on the course home page.

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1. Determine if each of the following statements is true or false. You must justify your answers. All matrices involved are assumed to be constant matrices unless otherwise specified.

(a) Consider $\dot{x} = A(t)x, \;\; x \in R^n$, where $A(t)$ is differentiable everywhere and $A\dot{A} = \dot{A}A$. Let $\Phi(t,s)$ be the state transition matrix. $\Phi(t,s) = e^{A(t)(t-s)}\ \forall t, s$, only if $A(t)$ is a constant matrix (Necessity. Sufficiency is trivial to show). (5p)

Answer: True, this can seen by taking $\dot{\Phi}(t,0)$.

(b) Consider $\dot{x} = Ax, y = Cx$, where $x \in R^n$. Let Ω denote the observability matrix. If $Cx(t) = 0, \ \forall t \geq 0$, then $\forall t \geq 0, \ x(t) \in ker\ \Omega$. .... (5p)

Answer: True, since $Cx(t) = 0, \ \forall t \geq 0$ implies $Cx^k(t) = CA^k x(t) = 0, \ \forall t \geq 0$.

(c) Suppose (A, b, c) is a minimal realization of a SISO transfer function $r(s) = \frac{n(s)}{d(s)}$, then $(c, A + bF)$ will still be observable for any F. .... (5p)

Answer: False. This is only true if $n(s)$ is a constant.

(d) Consider the dynamical Riccati equation

$$\begin{aligned} \dot{P} &= -A^T P - PA + PBR^{-1}B^T P - Q \\ P(t_1) &= S \end{aligned}$$

For any $t \in [t_0, t_1]$, the solution $P(t)$ is only positive semidefinite but not positive definite, if S is only positive semidefinite but not positive definite. (5p)

Answer: False, since $P(t)$ can be positive definite on $[t_0, t_1)$, for example, if Q is positive definite.

2. Consider :

$$\begin{aligned} \dot{x} &= Ax + bu \\ y &= cx, \end{aligned}$$

where

$$A = \begin{bmatrix} \alpha_1 & 1 \\ 0 & \alpha_2 \end{bmatrix}, \; b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \; c = \begin{bmatrix} 1 & 0 \end{bmatrix}, \text{and } \alpha_1, \alpha_2 \text{ are constants.}$$

(*a*) Find the state transition matrix e^{At}. .... (6p)
Answer: Omitted.

(*b*) When do the closed-loop poles of the system can be arbitrarily assigned by state feedback? .... (4p)
Answer: $\alpha_2 - \alpha_1 \neq 1$.

(*c*) Find an open-loop control $u(t)$ (i.e. $u(t)$ should be a function of time and the initial states) in any way you can, such that whenever $cx(0) = 0$, $y(t) = 0$ for all $t \geq 0$. .... (5p)
Answer: $u = -e^{(\alpha_2 - 1)t} x_2(0)$.

(*d*) What happens to $x_2(t)$ as $t \to \infty$ when the control designed in (c) is applied? (5p)
Answer: If $\alpha_2 - 1 < 0$, $x_2(t)$ converges to zero; if $\alpha_2 - 1 > 0$, $x_2(t)$ diverges.

3. Consider the transfer matrix

$$R(s) = \begin{bmatrix} \frac{\gamma}{s+1} & \frac{1}{s+1} & \frac{\gamma+2}{s+1} \\ \frac{1}{(s+1)(s+2)} & \frac{1}{(s+1)(s+2)} & \frac{3}{(s+1)(s+2)} \end{bmatrix},$$

where γ is a constant.

(*a*) Find the standard reachable realization. .... (6p)
Answer: Omitted.

(*b*) Compute the McMillan degree of $R(s)$. .... (5p)
Answer: $\delta(R) = 3$ if $\gamma \neq 1$, otherwise $\delta(R) = 2$

(*c*) For the case $\gamma = 1$, find a minimal realization of $R(s)$ and verify your answer if you use Kalman decomposition. .... (9p)
Answer: To find a minimal realization, it is enough to find the standard reachable realization for $\begin{bmatrix} \frac{1}{s+1} \\ \frac{1}{(s+1)(s+2)} \end{bmatrix}$.

4. Consider the optimal control problem

$$\min_u J = \int_0^\infty (y^2 + ku^2)\,dt$$

s.t.

$$\dot{x} = Ax + Bu$$
$$y = Cx$$
$$x(0) = x_0,$$

where, $k > 0$, and

$$A = \begin{bmatrix} a_1 & 1 \\ 0 & a_2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \end{bmatrix}.$$

Answer: This problem is almost the same as Problem 4 in the January exam, if we let $k = \epsilon^{-2}$. So the answers to (a), (b), (c) are omitted.

(a) Show for $a_1 = 0$, the associated algebraic Riccati equation (ARE) does not have a positive definite solution. . . . . (4p)

(b) Let $P(k)$ denote the symmetric solution to the ARE. Show that $\lim_{k\to\infty} k^{-1}P(k)$ is positive definite if and only if $a_1 > 0$ and $a_2 > 0$. . . . . (6p)

(c) Show when $\lim_{k\to\infty} k^{-1}P(k) > 0$, $\lim_{k\to\infty}(A - BB^T k^{-1}P(k))$ has eigenvalues $\{-a_1, -a_2\}$. . . . . (5p)

(d) When $\lim_{k\to\infty} k^{-1}P(k) > 0$, show for any $\sigma > 0$, there exists $k_0 > 0$, such that for all $k > k_0$, $A - \sigma BB^T P(k)$ is a stable matrix. . . . . (5p)

Answer: Making use the ARE associated with the optimal control problem, we can show $(A - \sigma BB^T P(k))^T P + P(A - \sigma BB^T P(k))$ is negative definite if k is sufficiently large.

5. (a) Consider a time-varying system $\dot{x} = A(t)x$. Recall that x_0 is called an equilibrium if $x(t, t_0) = x_0$ for all $t \geq t_0$, where $x(t, t_0)$ is the solution with $x(t_0, t_0) = x_0$. Show that $x_0 = 0$ is the unique equilibrium if $\|\Phi(t, t_0)\| \to 0$ as $t \to \infty$, where $\Phi(t, t_0)$ is the state transition matrix. Is this condition also necessary ("only if")? . . . . (6p)

Answer: The key in the proof is the fact $\Phi(t, t_0)x_0 = x_0$ if x_0 is an equilibrium. The condition is not necessary.

(b) This problem is related to the motion of a particle. Given a nonzero skew symmetric matrix A ($A^T = -A$, rotation), show that for almost all unit vectors $b \in R^3$ (translational motion) (A, b) is controllable, specifically show that there are only two unit vectors b that makes (A, b) uncontrollable. . . . . (8p)

Answer: Such an A matrix can always be determined by three parameters $a = (a_1, a_2, a_3)^T$. As long as b is not parallel or orthogonal to a, the pair will be controllable.

(c) Show that (A, C) is observable if and only if $(A, C^T C)$ is observable, where A is an $n \times n$ matrix and C is a $p \times n$ matrix. . . . . (6p)

Answer: Let Ω denote the observability matrix for (A, C), and $\bar{\Omega}$ for $(A, C^T C)$. We need to show $ker\ \Omega = ker\ \bar{\Omega}$. $ker\ \Omega \subset ker\ \bar{\Omega}$ is trivial to show. To show $ker\ \bar{\Omega} \subset ker\ \Omega$, we note that $x \in ker\ \bar{\Omega}$ iff $C^T C A^i x = 0,\ i = 0, 1, \cdots, n-1$. Then $x^T (A^i)^T C^T C A^i x = 0,\ i = 0, 1, \cdots, n-1$. Thus, $CA^i x = 0,\ i = 0, 1, \cdots, n-1$, namely $x \in ker\ \Omega$.

Good luck!