KTH Matematik

Solution to Exam January 14, 2016, SF2832 Mathematical Systems Theory.

We reserve the right to correct possible typing errors

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Allowed material: Anders Lindquist & Janne Sand, An Introduction to Mathematical Systems Theory, Per Enqvist, Exercises in Mathematical Systems Theory, your own class notes, and β mathematics handbook.

Solution methods: All conclusions should be carefully motivated.

Note! Your personal number must be stated on the cover sheet. Number your pages and write your name on each sheet that you turn in!

You need 45 points credit (including your still valid bonus) to pass the exam. The other grade limits are listed on the course home page.

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1. Determine if each of the following statements is true or false. You must justify your answers. All matrices involved are assumed to be constant matrices unless otherwise specified.

(a) Consider $\dot{x} = A(t)x,\ x \in R^n$. Let $\Phi(t,s)$ be the state transition matrix. If $\Phi^T(t,s) = \Phi(s,t)\ \forall t,s$, then $A(t)^T = -A(t)$. .... (5p)
Answer: True, since $\dot{\Phi}(t,s) = (\dot{\Phi}^T(t,s))^T = (\dot{\Phi}(s,t))^T = (-\Phi(s,t)A(t))^T = -A^T(t)\Phi(t,s)$.

(b) Consider $\dot{x} = Ax + bu$, where $x \in R^n,\ u \in R$. If all the eigenvalues of A are identical, then we can not find any b such that (A,b) is controllable. ..... (5p)
Answer: False, for example $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

(c) Consider a strictly proper transfer matrix $R(s)$. Let r denote the degree of the least common denominator of the elements of $R(s)$. Then

$$\text{rank } H_r \geq r,$$

where H_r is the block Hankel matrix defined in the compendium. .... (5p)
Answer: True, since $\delta(R) = rank\ H_r \geq r$.

(d) If (C,A) is observable, then the algebraic Riccati equation $A^TP+PA-PBB^TP+C^TC = 0$ always has a positive definite solution P, no matter what B is. (5p)
Answer: False, for example $B = 0$ and A is unstable.

2. Consider :

$$\begin{aligned} \dot{x} &= Ax + bu \\ y &= cx, \end{aligned}$$

where

$$A = \begin{bmatrix} 0 & -a_1 & 0 \\ a_1 & 0 & -a_2 \\ 0 & a_2 & 0 \end{bmatrix}, \; b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \; c = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix},$$

where a_1, a_2 are real numbers such that $a_1 a_2 \neq 0$.

(a) For the case $a_1^2 + a_2^2 = 1$ (this assumption is only for this sub-problem), calculate the state transition matrix e^{At} (hint: show first $A^{2i+1} = (-1)^i A,\ A^{2i} = (-1)^{i+1} A^2$, and trigonometric functions might be useful). ....(8p)
Answer: $e^{At} = I + A\sin(t) + A^2(1 - \cos(t))$.

(b) Propose a set of three eigenvalues for $A + bk$ such that $(c, A + bk)$ becomes unobservable, and show as long as $a_1 a_2 \neq 0$ those eigenvalues can actually be placed by some k (you do not have to give such a row vector k explicitly). (6p)
Answer: $r(s) = c(sI - A)^{-1} b = \frac{s^2 + a_1^2}{s(s^2 + a_1^2 + a_2^2)}$. So we need to choose eigenvalues to cancel at least one zero. When $a_1 a_2 \neq 0$, the system is controllable.

(c) Find a linear feedback control $u(t) = kx$ such that whenever $cx(0) = 0$, then $y(t) = 0$ for all $t \geq 0$. ....(3p)
Answer: $u = -a_2 x_2$ for example.

(d) Describe the trajectory $x(t)$ as $t \to \infty$ when the control designed in (c) is applied and the initial condition is such that $cx(0) = 0$?..................(3p)
Answer: $\dot{x}_1 = -a_1 x_2,\ \dot{x}_2 = a_1 x_1$, so the trajectory is a circle.

3. Consider the transfer matrix

$$R(s) = \begin{bmatrix} \frac{\gamma}{(s+1)^2} & \frac{1}{(s+1)(s+2)} \\ \frac{1}{(s+1)^2} & \frac{1}{(s+1)(s+2)} \\ \frac{1+\gamma}{(s+1)^2} & \frac{2}{(s+1)(s+2)} \end{bmatrix},$$

where γ is a constant.

(a) Find the standard reachable realization. .... (6p)
Answer: Omitted. Note that the third row is a linear combination of the first two.

(b) Compute the McMillan degree of $R(s)$. ....(5p)
Answer: $\delta(R) = 4$ if $\gamma \neq 1$ otherwise $\delta(R) = 3$.

(c) For the case $\gamma = 1$, find a minimal realization of $R(s)$ and verify your answer if you use Kalman decomposition...(9p)
Answer: We only need to find a minimal realization for $y = \begin{bmatrix} \frac{1}{(s+1)^2} & \frac{1}{(s+1)(s+2)} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \frac{1}{s+1} \begin{bmatrix} \frac{1}{(s+1)} & \frac{1}{(s+2)} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$. Then, $\dot{x}_1 = -x_1 + x_2 + x_3, \dot{x}_2 = -x_2 + u_1, \dot{x}_3 = -2x_3 + u_2, y = x_1$.

4. Consider the optimal control problem

$$\begin{aligned}\min_u J &= \int_0^\infty (\epsilon^2 y^2 + u^2)dt \\ \text{s.t.}& \\ \dot{x} &= Ax + Bu \\ y &= Cx \\ x(0) &= x_0,\end{aligned}$$

where, $\epsilon \neq 0$, and

$$A = \begin{bmatrix} a_1 & 1 \\ 0 & a_2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \end{bmatrix}.$$

(*a*) Show for $a_1 = 0$, the associated algebraic Riccati equation (ARE) does not have a positive definite solution. . . . . (6p)

Answer: Let $P = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}$. In this case one can easily show that $p_1 = p_2 = 0$.

(*b*) Let $P(\epsilon)$ denote the symmetric solution to the ARE. Show that $\lim_{\epsilon \to 0} P(\epsilon)$ is positive definite if and only if $a_1 > 0$ and $a_2 > 0$. . . . . (8p)

Answer: In this case one can rewrite the ARE as

$$-P^{-1}A^T - AP^{-1} = -BB^T.$$

Since $(-A, B)$ is controllable, P^{-1} is positive definite iff $-A$ is a stable matrix.

(*c*) Show when $\lim_{\epsilon \to 0} P(\epsilon) > 0$, $\lim_{\epsilon \to 0}(A - BB^T P(\epsilon))$ has eigenvalues $\{-a_1, -a_2\}$. (6p)

Answer: We have $p_3 = 2(a_1 + a_2)$, $p_2 = 2a_1(a_1 + a_2)$, and the result follows.

5. (*a*) Consider a one-dimensional system

$$\begin{aligned} x(t+1) &= ax(t) \\ y(t) &= x(t) + w(t), \end{aligned}$$

where $a \neq 0$,, both $x(0)$ and $w(t)$ are Gaussian with zero mean and covariances p_0 and σ respectively. respectively.

(*i*) Design a Kalman filter $\hat{x}(t)$ for $x(t)$. . . . . (2p)

(*ii*) Express the covariance matrix $p(t) = E\{(x(t) - \hat{x}(t))^2\}$ in terms of t, a, p_0, σ. (2p)

(*iii*) Show $|a - ak(t)| < 1$ as $t \to \infty$ (where $k(t)$ is the Kalman gain)? . . . . (2p)

Answer: omitted. This is a special case in Homework 3, 2013.

(*b*) Consider $\dot{x} = Ax, y = Cx$. Show that the system is observable if and only if the only solution that satisfies $Cx(t) \equiv 0$ $(\forall t \geq 0)$ is $x(t) \equiv 0$. . . . . (5p)

Answer: The system is observable iff $ker\ \Omega = 0$. On the other hand, $ker\ \Omega$ is A-invariant, i.e. if $x_0 \in ker\ \Omega$, then $e^{At}x_0 \in ker\ \Omega$. We can show easily that if $x(t) \in ker\ C$ for all $t \geq 0$ then $x(t) \in ker\ \Omega$. In this case since $x(t) = 0$ is the only such solution, we have $ker\ \Omega = 0$.

(*c*) In this problem we investigate the relative degree and zeros of single-input-single-output systems (SISO). Consider the state space representation

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t)\\ y(t) &= Cx(t)\end{aligned}$$

and its corresponding transfer function

$$G(s) = C(sI - A)^{-1}B = \frac{s^m + b_{m-1}s^{m-1} + \ldots + b_0}{s^n + a_1 s^{n-1} + \ldots + a_n}$$

where $A \in \mathbf{R}^{n\times n}$, $B \in \mathbf{R}^{n\times 1}$, $C \in \mathbf{R}^{1\times n}$. The relative degree $r = n - m$ is the excess degree of the denominator compared to the numerator. Show that

$$CA^kB = 0, \quad k = 0, 1, \ldots, r-2 \quad \text{and} \quad CA^{r-1}B \neq 0 \quad \cdots (5\text{p})$$

Answer: We can show this easily by Laurant expansion and the meaning of Markov parameters. In fact, we can show $CA^{r-1}B = 1$.

and $\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{r-1} \end{bmatrix}$ has full row rank. .... (4p)

Answer: Assume for some $k_i,\ i = 1, \cdots, r$, $\sum_{i=1}^{r} k_i CA^{i-1} = 0$, we need to show $k_i = 0,\ i = 1, \cdots, r$. This can be done by multiplying both sides of the equality first by B, thus $k_r = 0$, then AB and so forth.

Good luck!