KTH Matematik

Solution to Exam in SF2832 Mathematical Systems Theory
08.00-13.00, 16 January 2017

We reserve the right to correct typographical errors.

Examiner: Xiaoming Hu, tel. 790 7180.

Allowed material: Anders Lindquist & Janne Sand, An Introduction to Mathematical Systems Theory, Per Enqvist, Exercises in Mathematical Systems Theory, your own class notes, and β mathematics handbook.

Solution methods: All conclusions should be carefully motivated.

Note! Your personal number must be stated on the cover sheet. Number your pages and write your name on each sheet that you turn in!

You need 45 points credit (including your still valid bonus) to pass the exam. The other grade limits are listed on the course home page.

Read this before you start: 1. The problems are NOT ordered in terms of difficulty. 2. If the problem seems to be too complex (either in terms of calculation or abstraction), then it is likely that you have not found the best method yet.

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1. Determine if each of the following statements is true or false. You must justify your answers. All matrices involved are assumed to be constant matrices unless otherwise specified.

(a) Consider an n-dimensional time-varying system $\dot{x} = A(t)x$, where $A(t)$ is continuous. If $A^T(t) = -A(t)\ \forall t \in R$, then $\|x(t)\|^2 = \|x(t_0)\|^2\ \forall t \geq t_0$. ...... (5p)

Answer: True. Since $(\frac{\partial\Phi(t,s)}{\partial s})^T = A(s)\Phi(t,s)^T$, thus $\Phi(t,s)^T = \Phi(s,t)$, then $\|x(t)\|^2 = x(t_0)^T\Phi(t,s)^T\Phi(t,s)x(t_0) = \|x(t_0)\|^2\ \forall t \geq t_0$

(b) Consider $\dot{x} = Ax, y = Cx$, where $x \in R^n$. If $x_0 \notin ker\ \Omega$ (the unobservable subspace), then $y(t) = Ce^{At}x_0 \neq 0$ for any $t > 0$. .... (5p)

Answer: False, for example, $\dot{x}_1 = x_2, \dot{x}_2 = -x_1, y = x_1$.

(c) Assume (A, B) is controllable and (C, A) is observable. Then $(C, A + BK)$ is also observable for any K. .... (5p)

Answer: False, due to possible pole zero cancellation.

(d) Assume (A, B) is controllable. The algebraic Riccati equation $A^TP + PA - PBB^TP = 0$ has a positive definite solution P if and only if A is a stable matrix. .... (5p)

Answer: True. If P is positive definite, then the above equation is equivalent to a Lyapunov equation if we let $Q = P^{-1}$. (Note that a minus sign is missed. The subject in concern should be "-A". Thus, as the problem appears in the exam, the answer should be false).

2. Consider :

$$\dot{x} = Ax + bu$$

where

$$A = \begin{bmatrix} -1 & 1 & 1 \\ 0 & \alpha & 1 \\ 0 & 0 & 1 \end{bmatrix}, \ b = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \text{and } \alpha \neq 1 \text{ is constant.}$$

(a) Find the state transition matrix e^{At}. .... (6p)

Answer: Find first the state transition matrix for the subsystem consisting of the last two variables, then plug in $x_2(t), x_3(t)$ to the first equation. The rest is omitted.

(b) When is the pole placement problem solvable? .... (3p)

Answer: When the system is controllable, i.e., $\alpha \neq 0, -2$.

(c) Let $u = 0$. Find all solutions $x(t)$ that lie on the plane $D = \{x \in R^3 : [0\ 1\ 1]x = 0\}$. ....(5p)

Answer: Denote $c = [0\ 1\ 1]$, then all such solutions are solutions with initial conditions in $ker\ \Omega$. When $\alpha = 2$, $ker\ \Omega = span\{[1\ 0\ 0]^T, [0\ 1\ -1]^T\}$, otherwise $ker\ \Omega = span\{[1\ 0\ 0]^T\}$.

(d) Find $u(t) = Kx$ that makes D invariant, i.e., $[0\ 1\ 1]x(t) = 0, \forall t > 0$ if $[0\ 1\ 1]x(0) = 0$. .... (3p)

Answer: $u = -\frac{1}{2}(\alpha x_2 + 2x_3)$.

(e) Do the solutions on D converge to the origin as $t \to \infty$ when the control designed in (d) is applied? .... (3p)

Answer: When $\alpha < 0$.

3. Consider the transfer matrix

$$R(s) = \begin{bmatrix} \frac{\alpha}{s+2} & \frac{2}{s+2} \\ \frac{2}{s+\beta} & \frac{1}{s+2} \end{bmatrix},$$

where α, β are real nonzero constants.

(a) Determine the McMillan degree of $R(s)$. .... (8p)

Answer: $\delta(R) = 3$ if $\beta \neq 2$. When $\beta = 2$, $\delta(R) = 2$ if $\alpha \neq 4$; otherwise $\delta(R) = 1$.

(b) Find the standard reachable realization. .... (8p)

Answer: One should consider two cases: $\beta = 2$ and $\beta \neq 2$. The detail is omitted.

(c) When is the realization in (b) also observable? .... (4p)

Answer: When $\delta(R) = 2$.

4. Consider the optimal control problem

$$\min_u J = \int_0^\infty (x^T Q x + u^2)dt \quad \text{s.t.} \quad \dot{x} = Ax + Bu,\ x(0) = x_0,$$

where,

$$A = \begin{bmatrix} k & 1 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad Q = \begin{bmatrix} 0 & 0 \\ 0 & q^2 \end{bmatrix}.$$

(a) Show that for $k = 0$, the associated algebraic Riccati equation (ARE) does not have a positive definite solution. .... (4p)

Answer: Let $P = \begin{pmatrix} p_1 & p_2 \\ p_2 & p_3 \end{pmatrix}$, then $p_1 = p_2 = 0$.

(b) Now suppose $q = 0$. What is the condition on k such that the ARE has a positive definite solution? .... (8p)

Answer: $-A$ must be a stable matrix, thus $k > 0$.

(c) Again suppose $q = 0$. Show that when the ARE has a positive definite solution, the closed-loop system has poles $\{-k, -1\}$. .... (4p)

Answer: From the ARE, we have $A - BB^T P = -P^{-1} A^T P$.

(d) When $q = 0$, what is the condition on k such that the optimal control exists? (4p)

Answer: $k \neq 0$.

5. (a) Let $C = \begin{bmatrix} C_1 & 0 \end{bmatrix}$ and $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$, where C_1 is an $k \times n_1$ matrix with rank n_1. The matrices A_{12} and A_{22} have dimensions $n_1 \times (n - n_1)$ and $(n - n_1) \times (n - n_1)$ respectively. Show that (C, A) is observable if and only if (A_{12}, A_{22}) is observable. .... (5p)

Answer: Clearly $C_1 x_1(t) = 0\ \forall t \geq 0$ iff $x_1 = 0\ \forall t \geq 0$, thus $\dot{x}_1 = 0$. Therefore $A_{12} x_2 = 0$. The whole system is observable iff that $A_{12} x_2 = 0$ implies $x_2 = 0$, i.e., (A_{12}, A_{22}) is observable.

(b) Let (A, B) be controllable and consider the equation

$$AP + PA^T + BB^T + Q = 0, \tag{1}$$

where $Q \geq 0$. This equation is slightly different from that in Corollary 4.3.6 of the compendium.

Show that the following statements are equivalent

(a) A is a stable matrix

(b) the equation has a positive definite solution P.

(Hint: try to apply Corollary 4.3.6) .... (5p)

Answer: Let $\bar{B}\bar{B}^T = BB^T + Q$, then $(A, \bar{B})$ is also controllable. Then we can use Corollary 4.3.6.

(c) Consider a controllable system $\dot{x} = Ax + Bu$. Given an initial state x_0, using Bellman's principle (or as is shown in Homework 3) we know that the minimum energy control for reaching $x(t_1) = 0$ can be written in feedback form as

$$u = -B^T e^{A^T(t_1-t)} W(t, t_1)^{-1} e^{A(t_1-t)} x(t) \triangleq -B^T P(t, t_1) x(t),$$

where $W(t, t_1)$ is the reachability Gramian.

Show that if we use the constant gain feedback control $u = -B^T P(0, t_1) x(t)$ where t_1 is any positive constant, then $\lim_{t\to\infty} x(t) = 0$, $\forall x_0 \in R^n$. (Hint: show first $-AP(0, t_1)^{-1} - P(0, t_1)^{-1} A^T + BB^T = e^{-At_1} BB^T e^{-A^T t_1}$) ....(10p)

Answer: Denote $P(0, t_1) = e^{A^T t_1} W(0, t_1)^{-1} e^{At_1}$, then $P^{-1}(0, t_1) = \int_0^{t_1} e^{-As} BB^T e^{-A^T s} ds$. Let $Q = e^{-At} BB^T e^{A^T t}$, then $\dot{Q} = -AQ - QA^T$. Thus, $e^{-At_1} BB^T e^{-A^T t_1} - BB^T = -AP^{-1} - P^{-1} A^T$, which leads to $(A - BB^T P(0, t_1)) P^{-1} + P^{-1} (A - BB^T P(0, t_1))^T = -BB^T - e^{-At_1} BB^T e^{-A^T t_1}$.

Good luck!