KTH Matematik

Solution to Exam January 9, 2018 in SF2832 Mathematical Systems Theory.

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Allowed books: Course compendium by Anders Lindquist et. al, Exercise notes by Per Enqvist, your own hand-written class notes, and β mathematics handbook.

Solution methods: All conclusions should be carefully motivated.

Note! Your personal number must be stated on the cover sheet. Number your pages and write your name on each sheet that you turn in!

You need 45 points credit (including your still valid bonus) to pass the exam. The other grade limits are listed on the course homepage.

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1. Determine if each of the following statements is true or false. You must justify your answers. All matrices involved are assumed to be constant matrices unless otherwise specified.

(a) Consider $\dot{x} = Ax$ and assume $Ax_1 = 0$, where $A \neq 0$ and $x_1 \neq 0$. Then there exists a solution $x(t)$ such that $x(0) \neq x_1$ and $x(T) = x_1$ for some finite $T > 0$. (5p)

Answer: False, otherwise $x(0) = e^{-AT}x_1 = x_1$ (by Taylor expansion).

(b) Consider a single input system $\dot{x} = Ax + bu$ where $x \in R^n$. If rank $A < n - 1$, then the system is not controllable. .... (5p)

Answer: True, since a controllable system can be put into the canonical controllable form in which rank A is at least $n - 1$.

(c) Given a minimal realization (A, B, C), $(A+BLC, B, C)$ is also minimal for any L. .... (5p)

Answer: True, since output feedback does not change observability either (dual to that state feedback does not change controllability).

(d) The algebraic Riccati equation $A^TP + PA - PBB^TP + C^TC = 0$ has a positive definite solution P if and only if (A, B, C) is a minimal realization. ...... (5p)

Answer: False. For example, $B = 0$, (C, A) observable and A is a stable matrix.

2. Consider a three dimensional SISO system:

$$\begin{aligned} \dot{x} &= Ax + bu \\ y &= cx, \end{aligned}$$

namely $x \in R^3, u \in R, y \in R$. Assume A is given and $A^T = -A \neq 0$.

(a) Determine $a \in R^3$ that depends on the elements of A such that $Ax = a \times x$ for all $x \in R^3$, where "$\times$" denotes vector cross product. .... (2p)

Answer: Such an A can always be expressed as

$$A = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix},$$

then $a = (a_1\ a_2\ a_3)^T$.

(b) Assume $|a| = 1$, show that

$$e^{At} = I + A\sin(t) + A^2(1 - \cos(t)).$$

(Hint: show first $A^{2n+1} = (-1)^n A,\ n \geq 0$ and $A^{2n} = (-1)^{n+1} A^2,\ n \geq 1$.) (7p)

Answer: The expression is obtained by doing Taylor expansion on e^{At}.

(c) Find geometric conditions (in terms of the angle between two vectors) on a, b and c^T such that (A, b, c) is a minimal realization. .... (4p)

Answer: Since $Ab = a \times b,\ A^2 b = a \times (a \times b)$, we conclude that vectors a and b can neither be orthogonal nor parallel. We can draw similar conclusion on a and c^T.

(d) Let $x(0) = a$, compute $e^{At}x(0)$. .... (3p)

Answer: $e^{At}a = 0$ since $a \times a = 0$.

(e) Show $A - bb^T$ is a stable matrix if (A, b) is controllable. .... (4p)

Answer: Let $P = \frac{1}{2}I$, then $(A - bb^T)P + P(A - bb^T)^T = -bb^T$. Since $(A - bb^T, b)$ is also controllable, the conclusion follows.

3. Consider the transfer matrix

$$R(s) = \begin{bmatrix} \frac{\alpha}{s+2} & \frac{2}{s+2} \\ \frac{2}{s+\beta} & \frac{\gamma}{s+2} \end{bmatrix},$$

where α, β, γ are real constants.

(a) Find the standard reachable realization. .... (8p)

Answer: One should consider the cases $\beta = 2$ and $\beta \neq 2$. The rest is omitted.

(b) Determine the McMillan degree of $R(s)$. .... (6p)

Answer: 1) $\beta \neq 2$, $\delta(R) = 3$. 2) $\beta = 2$, a. $\alpha\gamma \neq 4$, $\delta(R) = 2$, b. $\alpha\gamma = 4$, $\delta(R) = 1$.

(c) When is the realization in (a) also observable? .... (2p)

Answer: $\beta = 2$ and $\alpha\gamma \neq 4$.

(d) Find a minimal realization for $R(s)$ for the case $\alpha = \beta = \gamma = 2$. .... (4p)

Answer: This can be reduced to realizing $\frac{2}{s+2}$, which is omitted.

4. Consider a state space system (A,b) as follows

$$\dot{x}_1 = a_1 x_1 + a_2 x_2$$
$$\dot{x}_2 = -a_2 x_1 + u,$$

where $a_1,\ a_2$ are real constants, and the cost function is

$$J = \int_0^{t_1} (x_2^2 + \epsilon^2 u^2) dt,$$

where $\epsilon > 0$. Assume $x^*(t)$ is the optimal trajectory for a given initial point $(x_1(0)\ x_2(0))^T$ with the optimal control $u = -\epsilon^{-2} b^T P(t) x^*(t)$.

(a) Compute $P(t)$ for the case $a_2 = 0$. .... (5p)
Answer: $p_1(t) = p_2(t) = 0$, $p_3(t) = \epsilon + z(t)^{-1}$, where $z(t) = -0.5\epsilon^{-1}(1 + e^{-2\epsilon^{-1}(t-t_1)}$.

(b) For what $a_1,\ a_2$ is $P(t)$ positive definite $\forall t < t_1$? .... (5p)
Answer: $a_2 \neq 0$.

(c) Assume $a_1 \neq 0$. Now let $t_1 = \infty$ and $P_\infty(\epsilon)$ be the positive semi-definite solution of the algebraic Riccati equation. For the case where $P_\infty(\epsilon)$ is positive definite, compute the finite eigenvalues of $A - \epsilon^{-2} bb^T P_\infty(\epsilon)$ as $\epsilon \to 0$. .... (10p)
Answer: Let $\bar{p}_2 = \epsilon^{-1} p_2,\ \bar{p}_3 = \epsilon^{-1} p_3$. Then, as $\epsilon \to 0$, Case 1: $a_1 < 0$, $p_1 = \bar{p}_2 = 0,\ \bar{p}_3 = 1$, thus the finite eigenvalue is a_1; Case 2: $a_1 > 0$, $p_1 = \frac{2a_1}{a_2^2}, \bar{p}_2 = \frac{2a_1}{a_2}, \bar{p}_3 = 1$, thus the finite eigenvalue is $-a_1$.

5. In this problem you are required to prove a basic result on observability and to solve an estimation problem using Kalman filter.

(a) Let $Q = C^T C$. Show that (A, Q) is observable if and only if (A, C) is observable, where A is an $n \times n$ matrix and C is a $p \times n$ matrix. .... (10p)
Answer: Let $\Omega_Q = (Q^T\ A^T Q^T \cdots (A^T)^{n-1} Q^T)^T$, we need to show ker $\Omega_Q =$ $ker\ \Omega$. Suppose $z \in ker\ \Omega_Q$, i.e. $\Omega_Q z = 0$. Then $C^T C A^k z = 0$, for all k. Thus, $z^T (A^k)^T C^T C A^k z = \|CA^k z\|^2 = 0$, then $CA^k z = 0$. Thus, $ker\ \Omega_Q \subset ker\ \Omega$. It is even easier to show $ker\ \Omega \subset ker\ \Omega_Q$.

(b) Let x be a scalar parameter we do not have any priory knowledge about. We would like to estimate the value of x by a set of noisy measurements

$$y_i = x + w_i \text{ for } i = 1, 2, \dots, n$$

where $w_i \in N(0, \sigma_i^2)$ are independent of each other and of x. Determine $\hat{x}_n$ and $P_n = E\{(x - \hat{x}_n)^2\}$ as a function of $\sigma_1, \cdots, \sigma_n$, where $\hat{x}_n$ is the optimal estimate of x based on the n measurements, in the sense that the mean square error is minimized. .... (10p)

Answer: We can take the model $x(t+1) = x(t)$ and consider all the measurements are taken at $t = 0$. Then we have $C = [1 \cdots 1]^T$, $Q = 0$ and $R = diag\{\sigma_1^2, \cdots, \sigma_n^2\}$. Clearly, $P(0)^{-1} = P_0^{-1} = 0$ since we do not have any priory knowledge, thus $\hat{x}(0) = 0$. Then $\hat{x}_n = \hat{x}(1) = K(0)y = P_0C^T(CP_0C^T + R)^{-1}y = (C^TR^{-1}C)^{-1}C^TR^{-1}y$. $P_n = P(1) = P_0 - K(0)CP_0 = (C^TR^{-1}C)^{-1}$.