KTH Matematik

Solution to Exam in SF2832 Mathematical Systems Theory
14.00-19.00, January 8, 2020

Examiner: Xiaoming Hu, tel. 790 7180.

Allowed material: Anders Lindquist & Janne Sand, An Introduction to Mathematical Systems Theory, Per Enqvist, Exercises in Mathematical Systems Theory, your own class notes, and β mathematics handbook.

Solution methods: All conclusions should be carefully motivated.

Note! Your personal number must be stated on the cover sheet. Number your pages and write your name on each sheet that you turn in!

You need 45 points credit (including your still valid bonus) to pass the exam. The other grade limits are listed on the course home page.

Read this before you start: 1. The problems are NOT ordered in terms of difficulty. 2. If the problem seems to be too complex (either in terms of calculation or abstraction), then it is likely that you have not found the best method yet.

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1. Determine if each of the following statements is true or false. You must justify your answers. All matrices involved are assumed to be constant matrices unless otherwise specified.

(a) Consider an n-dimensional *time-varying* system $\dot{x} = A(t)x$, where $A(t)$ is continuous. If $A^T(t) = -A(t)\ \forall t \in R$, then $\Phi^T(t,s) = \Phi^{-1}(t,s)$, where $\Phi(t,s)$ is the state transition matrix. .... (5p)

Answer: True. This can be seen using the property $\frac{\partial}{\partial s}\Phi(t,s) = -\Phi(t,s)A(s)$, thus $\frac{\partial}{\partial s}\Phi^T(t,s) = -A^T(s)\Phi^T(t,s) = A(s)\Phi^T(t,s)$.

(b) Consider $\dot{x} = Ax + bu$, where $x \in R^n,\ u \in R$. If $rank\ A < n-1$, then (A,b) is never controllable. .... (5p)

Answer: True. Since a controllable system can always be transformed into the canonical controllable form in which A has rank at least $n-1$.

(c) Given a strictly proper rational matrix function $R(s)$, if the dimension of its standard reachable realization is equal to that of its standard observable realization, then that dimension must be equal to the McMillan degree of $R(s)$. (5p)

Answer: False. One can easily find a counter example in the compendium.

(d) Consider the Riccati differential equation:

$$\begin{aligned} \dot{P}(t) &= -A^TP(t) - P(t)A + P(t)BB^TP(t) - C^TC \\ P(t_1) &= P_1, \end{aligned}$$

where C is a $p\times n$ matrix and $p < n$. If P_1 is only positive semidefinite ($det\ P_1 = 0$), then $det\ P(t) = 0$ for any $t < t_1$. ....(5p)

Answer: False. For example, when (C,A) is observable.

2. Consider :

$$\begin{aligned}\dot{x} &= Ax + bu \\ y &= cx,\end{aligned}$$

where

$$A = \begin{bmatrix} \alpha & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \; b = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \; c = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}, \text{ and } \alpha \text{ is constant.}$$

(a) Find the state transition matrix e^{At}. .... (6p)
Answer: Solve first the second equation for obtaining $x_2(t) = e^{-t}x_{20}$, then plug in $x_2(t)$ to the third equation as input to obtain $x_3(t) = e^t x_{30} + \frac{1}{2}(e^t - e^{-t})x_{20}$, then plug in $x_3(t)$ to the first equation to obtain $x_1(t) = e^{\alpha t}x_{10} + \frac{1}{1-\alpha}(e^{(1-\alpha)t} - 1)x_{30} + (\frac{1}{1-\alpha}(e^{(1-\alpha)t} - 1) - \frac{1}{1+\alpha}(e^{(1+\alpha)t} - 1))x_{20}$. For $\alpha = 1$ or $\alpha = -1$, the relevant terms should be understood as the limit approaching the respective value of α. By expressing $x(t) = e^{At}x_0$, we obtain e^{At}.

(b) For what α is the pole placement problem solvable? .... (3p)
Answer: $\alpha \neq 0$. This is when the system is controllable.

(c) Let $u = 0$. Find all solutions $x(t)$ that lie $\forall t \geq 0$ on the plane $\mathcal{D} = \{x \in R^3 : cx = 0\}$. .... (5p)
Answer: $cx(t) = x_2(t)+x_3(t) = 0 \; \forall t \geq 0$ implies that $\dot{x}_2(t)+\dot{x}_3(t) = x_3(t) = 0$. Then $x_2(t) = x_3(t) = 0 \; \forall t \geq 0$. From the first equation we have $x_1(t) = e^{\alpha t}x_1(0)$.

(d) Find $u(t) = Kx$ that makes $\mathcal{D}$ invariant, i.e., $cx(t) = 0, \forall t \geq 0$ if $cx(0) = 0$. (3p)
Answer: Again $cx(t) = x_2(t) + x_3(t) = 0 \; \forall t \geq 0$ implies that $\dot{x}_2(t) + \dot{x}_3(t) = x_3(t) + u(t) = 0$, thus $u = -x_3$ or $u = x_2$ (since $x_2(t) + x_3(t) = 0$).

(e) For what α is every solution $x(t)$ in $\mathcal{D}$ bounded when the control designed in (d) is applied? .... (3p)
Answer: By keeping $x_2(t) + x_3(t) = 0$ or $x_3(t) = -x_2(t)$ and using $u = -x_3$ the system becomes $\dot{x}_1 = \alpha x_1, \; \dot{x}_2 = 0$, then $\alpha \leq 0$ will make all the solutions bounded.

3. Consider the transfer matrix

$$R(s) = \begin{bmatrix} \frac{1}{s+\beta} & \frac{1}{s+\beta} \\ \frac{1}{s+2} & \frac{\alpha}{s+2} \end{bmatrix},$$

where α, β are real constants.

(a) Determine the McMillan degree of $R(s)$. .... (8p)
Answer: If $\beta = 2$ and $\alpha = 1$, we have four elements of $R(s)$ as all the minors. Their least common denominator is $s + 2$, thus $\delta(R) = 1$. Otherwise we have either an order 2 minor $\frac{1}{(s+2)(s+\beta)}$ (if $\alpha \neq 1$) or $\beta \neq 2$ with four order 1 minors, in either case their least common denominator is $(s + 2)(s + \beta)$, thus$\delta(R) = 2$.

(b) Find the standard reachable realization for the case $\beta \neq 2$. . . . . (4p)
Answer: We have $m = k = 2$ and $\chi(s) = (s+\beta)(s+2) = s^2 + (2+\beta)s + 2\beta$, then

$$\chi(s)R(s) = \begin{bmatrix} 2 & 2 \\ \beta & \alpha\beta \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & \alpha \end{bmatrix} s.$$

Using the expression on p.40 in the compendium we obtain the realization.

(c) For the case $\alpha = 1,\ \beta \neq 2$ find a minimal realization and verify your answer by checking if $C(SI - A)^{-1}B = R(s)$. . . . . (8p)
Answer: Since the two columns of $R(s)$ are the same, we can first do a controllable realization only for u_1, then replace u_1 by $u_1 + u_2$ in the state space model. Then we have $m = 2,\ k = 1$ and $\chi(s) = (s+\beta)(s+2) = s^2 + (2+\beta)s + 2\beta$, then

$$\chi(s)R(s) = \begin{bmatrix} 2 \\ \beta \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} s.$$

We have

$$A = \begin{bmatrix} 0 & 1 \\ -2\beta & -(2+\beta) \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ C = \begin{bmatrix} 2 & 1 \\ \beta & 1 \end{bmatrix}.$$

Then we should change to

$$B = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}.$$

4. Consider the optimal control problem

$$\min_u J = \int_0^{t_1} u^T u dt + x(t_1)^T S x(t_1) \quad \text{s.t.} \quad \dot{x} = Ax + Bu,\ x(0) = x_0,$$

where (A, B) is controllable and S is positive definite.

Let $u = -B^T P(t_1 - t)x$ denote the optimal control.

(a) Solve the Riccati equation to obtain $P(t_1 - t)$ by solving the adjoint system. (**Hint:** to determine P is the same as determining P^{-1} if P is invertible) (10p)
Answer: By using the adjoint system, we have $Y = exp(A^T(t_1 - t))S$, $X = exp(-A(t_1 - t)) + \int_0^{t_1 - t} exp(-As)BB^T exp(-A^T s) ds exp(A^T(t_1 - t))S$.
$P^{-1} = XY^{-1} = exp(-A(t_1 - t))S^{-1}exp(-A^T(t_1 - t)) + \int_0^{t_1 - t} exp(-As)BB^T exp(-A^T s) ds$.

(b) Compute $\lim_{t_1 - t \to \infty} P(t_1 - t)$ for the case A is a stable matrix. . . . . (4p)
Answer: If A is stable, then $Y \to 0$ and $X \to \infty$, thus $P^{-1} \to \infty$.

(c) What are the eigenvalues of $\lim_{t_1 - t \to \infty}(A - BB^T P(t_1 - t))$ for the case $-A$ is a stable matrix? . . . . (6p)
Answer: If $-A$ is stable, $P^{-1} \to \int_0^{\infty} exp(-As)BB^T exp(-A^T s) ds$, which satisfies $-P^{-1}A^T - AP^{-1} + BB^T = 0$. Thus, $A - BB^T P = -P^{-1}A^T P$, which has same eigenvalues as $-A^T$ thus as $-A$.

5. (a) Let $C = \begin{bmatrix} C_1 & 0 \end{bmatrix}$ and $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$, where C_1 is an $k \times n_1$ matrix with rank n_1. The matrices A_{12} and A_{22} have dimensions $n_1 \times (n - n_1)$ and $(n - n_1) \times$

$(n - n_1)$ respectively. Show that (C, A) is observable if and only if (A_{12}, A_{22}) is observable. ....(6p)

Answer: (C, A) is observable iff $Cx(t) = 0\ \forall t \geq 0$ implies $x(t) = 0$. $C_1x_1(t) = 0$ implies $x_1(t)^T C_1^T C_1 x_1(t) = 0$, which implies $x_1(t) = 0$ since C_1 has full column rank. Then $A_{12}x_2(t) = 0$, where $\dot{x}_2(t) = A_{22}(t)$. $A_{12}x_2(t) = 0$ implies $x_2(t) = 0$ iff (A_{12}, A_{22}) is observable.

(b) Consider the algebraic Riccati equation

$$A^T P + PA - PBB^T P + C^T C = 0.$$

(1) Assume P is a real positive **semidefinite** solution. Show that $kerP$ is A-invariant (i.e, $\forall x \in kerP,\ Ax \in kerP$) and $kerP \subset kerC$. ....(4p)

Answer: Suppose $x \in KerP$. Multiplying both sides of the ARE by x:

$$PAx + C^T Cx = 0,$$

similarly $x^T C^T Cx = 0$. Therefore $x \in KerC$, which implies $kerP \subset kerC$. Furthermore, this leads to that $PAx = 0$. Thus $kerP$ is A-invariant.

(2) Show that if (C, A) is observable, then every positive semidefinite solution P is positive definite. **Hint:** use the conclusions in (1) ....(4p)

Answer: When (C, A) is observable, the only A-invariant subspace in $KerC$ (unobservable subspace) is $\{0\}$. Thus, $kerP = \{0\}$.

(c) Consider Kalman filter for discrete time linear systems as defined in Section 9.1 of the compendium. All the notations used below are also defined in Section 9.1, in particular, $P(t) = E\{(x(t) - E^{H_{t-1}(y)}x(t))(x(t) - E^{H_{t-1}(y)}x(t))^T\}$ and $P_t(t) = E\{(x(t) - E^{H_t(y)}x(t))(x(t) - E^{H_t(y)}x(t))^T\}$. Show that

$$P_t(t) = P(t) - P(t)C^T[CP(t)C^T + DRD^T]^{-1}CP(t),$$

assuming that the matrix inverse exists. ....(6p)

Answer: $P_t(t) = E\{(x(t) - E^{H_t(y)}x(t))(x(t) - E^{H_t(y)}x(t))^T\} = E\{(x(t) - E^{H_{t-1}(y)}x(t) - K\tilde{y})(x(t) - E^{H_{t-1}(y)}x(t) - K\tilde{y})^T\} = E\{(\tilde{x} - K\tilde{y})(\tilde{x} - K\tilde{y})^T\}$. $E\{(\tilde{x} - K\tilde{y})(\tilde{x} - K\tilde{y})^T\} = P(t) - KE(\tilde{y}\tilde{x}^T) - E(\tilde{x}\tilde{y}^T)K^T + KE(\tilde{y}\tilde{y}^T)K^T = P(t) - KCP(t) - P(t)C^T K^T + K(CP(t)C^T + DRD^T)K^T = P(t) - KCP(t)$. Plug in K the conclusion follows.