KTH Matematik

Solution to Exam in SF2832 Mathematical Systems Theory
Part one: 8:00-10:00, April 7, 2021

Examiner: Xiaoming Hu, tel. 0707967831.

Important! This exam consists of two parts. The second part starts at 10:20 and you will receive questions for Part two after the exam on Part one is concluded. You write the solutions on paper and then upload the scanned (or photoed) solutions in pdf format to Canvas. You must upload the solutions to Part one before Part two of the exam starts!

Allowed material: Anders Lindquist & Janne Sand, An Introduction to Mathematical Systems Theory (pdf or paper version), Per Enqvist, Exercises in Mathematical Systems Theory (pdf or paper version), your own class notes (digital or paper version), and β mathematics handbook.

Note! Your personal number must be stated on the cover sheet. Number your pages and write your name on each sheet that you turn in!

You need 40 points credit to pass the exam. The other grade limits are listed on the course home page.

Matrix notation: We use $A(t)$ to denote a time-varying matrix and A to denote a constant matrix.

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1. (20p) Determine if each of the following statements is true or false. **You must motivate your answers. No motivation no point.**

(a) Consider an n-dimensional system $\dot{x}(t) = Ax(t)$. If $\lim_{t\to\infty} \|e^{At}\| = \infty$, then A has at least one eigenvalue with positive real part. . . . . (6p)

Answer: False. A counter example is the double integrator.

(b) Consider $\dot{x} = Ax + Bu,\ y = Cx$, where $x \in R^n,\ u \in R,\ y \in R$.

b1. If $rank\ A < n-1$, then (C, A) can never be observable. . . . . (3p)

Answer: True, since (C, A) being observable is the same as (A^T, C^T) being controllable, which implies that A^T must have rank at least $n-1$.

b2. If (A, B) is controllable, then we can always find a C such that (C, A) is observable. . . . .(3p)

Answer: True, after converting the system into the canonical controllable form, we can choose for example $C = (1\ 0\ \cdots\ 0)$.

(c) Consider a strictly proper transfer matrix $R(s)$. Let r denote the degree of the least common denominator of the elements of $R(s)$. Then $\delta(R) \geq r$, where $\delta(R)$ is the McMillan degree of $R(s)$. . . . . (4p)

Answer: True, since for any minimal realization $deg\ \rho(s) = deg\ det(sI - A) = \delta(R)$, and $\rho(s)$ is a common denominator of the elements of $R(s)$.

(d) Suppose A is a stable matrix (all eigenvalues of A have negative real parts). Then, for any positive definite matrix P, $-(A^TP + PA)$ is at least positive semi-definite. .... (4p)
Answer: False, a counter example is $P = I$ and $A = (0\ 2; -1\ \ -1)$.

2. (25p) Consider :

$$\begin{aligned} \dot{x} &= Ax + bu \\ y &= cx, \end{aligned} \tag{1}$$

where $x \in R^3,\ u \in R,\ y \in R$, and the transfer function is

$$r(s) = c(sI - A)^{-1}b = \frac{s^2 + \alpha s + 1}{(s^2+1)(s+1)},$$

where α is a constant. Assume $c = \begin{bmatrix} 1 & \alpha & 1 \end{bmatrix}$.

(a) Find matrices A and b. .... (5p)
Answer: $A = (0\ 1\ 0; 0\ 0\ 1; -1\ \ -1\ \ -1),\ b = (0\ 0\ 1)^T$.

(b) Is your (A, b) reachable? .... (2p)
Answer: Yes.

(c) Does $\lim_{t\to\infty} e^{At}x_0 = 0$ for all $x_0 \in R^3$? .... (2p)
Answer: No, since A has two eigenvalues on the imaginary axis.

(d) For what α is (c, A) NOT observable? .... (6p)
Answer: $\alpha = 0, 2$.

(e) Find $u(t) = Kx$ that makes $\mathcal{D} = \{x \in R^3 : cx = 0\}$ invariant, i.e., $cx(t) = 0, \forall t \geq 0$ if $cx(0) = 0$. .... (3p)
Answer: $u(t) = Kx$ should make $\dot{y} \equiv 0$, which leads to $u = x_1 + (1-\alpha)x_3$.

(f) For $\alpha = -2$, give explicit expression for the state trajectories $x(t)$ obtained in (e). A solution $x(t)$ obtained in (e) means a solution to (1) when $u(t) = Kx$ is used and $cx(0) = 0$. .... (5p)
Answer: With the constraint $y = 0$ the system is reduced to $\dot{x}_1 = x_2,\ \dot{x}_2 = -x_1 + 2x_2$. Then all such solutions can be expressed as $x(t) = e^{Nt}x_0$, where $e^{Nt} = (e^t - te^t\ te^t; -te^t\ e^t + te^t)$.

(g) For which α you find in (d) can we find a two dimensional minimal realization of the $r(s)$ such that for the minimal realization there exists $u(t) = Kx$ that makes both $cx(t) = 0, \forall t \geq 0$ if $cx(0) = 0$ and $\lim_{t\to\infty} x(t) = 0$. .... (2p)
Answer: $\alpha = 2$.