KTH Matematik

Solution to Exam in SF2832 Mathematical Systems Theory
8.00-13:00, April 20, 2022

Examiner: Xiaoming Hu, tel. 0707967831.

Allowed material: Anders Lindquist & Janne Sand, An Introduction to Mathematical Systems Theory, Per Enqvist, Exercises in Mathematical Systems Theory, your own class notes, and β mathematics handbook.

Note! Your personal number must be stated on the cover sheet. Number your pages and write your name on each sheet that you turn in!

You need 40 points credit to pass the exam. The other grade limits are listed on the course home page.
The sub-problems in each problem are listed by the ascending order of difficulty whenever it is possible.

Matrix notation: We use $A(t)$ to denote a time-varying matrix and A to denote a constant matrix.

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1. (20p) Determine if each of the following statements is true or false. **You must motivate your answers. No motivation no point.**

(*a*) Consider a time-varying linear system $\dot{x} = A(t)x,\ x(t_0) = x_0$, where $x \in R^n$ and assume $A^T(t) = -A(t)\ \forall t$, then $\|x(t)\| = \|x_0\|\ \forall t \geq t_0$ and $\forall x_0 \in R^n$. (6p)
Answer: True, since $\frac{d}{dt}\|x(t)\|^2 = 0$.
For all the following sub-problems we consider

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx, \end{aligned}$$

where $x \in R^n\ (n > 1),\ u \in R^m,\ y \in R^p$.

(*b*) b1. Assume that there exists $x_0 \neq 0$ such that $Ax_0 = 0$. Then (C, A) is not observable. ....(3p)
Answer: False. A counter example is double integrator with the first state as output.
b2. Assume that both A and B have full column rank. Then (A, B) is reachable. (3p)
Answer: False, for example, $A = I$.

(*c*) Assume A is a stable matrix (all eigenvalues with negative real parts), then for any positive definite matrix P, $-(A^TP + PA)$ is either positive definite or positive semi-definite. ....(4p)
Answer: False. A counter example is $A = [0\ 2; -1\ \ -1],\ P = I$.

(*d*) Consider the optimal control problem for the system with $x(0) = x_0$:

$$\min_u \int_0^\infty (x^T C^T C x + u^T R u) dt,$$

where $R > 0$ and assume that (A, B) is controllable. If (C, A) is not observable, then the optimal control does not exist. .... (4p)

Answer: False. A counter example is that $C = 0$ and A is a stable matrix.

2. (25p) Consider :

$$\begin{aligned} \dot{x} &= Ax + bu \\ y &= cx, \end{aligned} \tag{1}$$

where $x \in R^3,\ u \in R,\ y \in R$, and the transfer function is

$$r(s) = c(sI - A)^{-1} b = \frac{s^2 + c_2 s + c_1}{(s^2 - 1)(s - 2)},$$

where $c_1,\ c_2$ are constant. Assume $b = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$.

(*a*) Find matrices A and c. .... (5p)

Answer: $c = [c_1\ c_2\ 1]$. Since $(s^2 - 1)(s - 2) = s^3 - 2s^2 - s + 2$, we have $A = [0\ 1\ 0; 0\ 0\ 1; -2\ 1\ 2]$.

(*b*) Is $\|e^{At} x_0\|$ less or equal to $M\|x_0\|$ for all $x_0 \in R^3$ and all $t \geq 0$? where M is a positive constant. .... (4p)

Answer: No since both 1 and 2 are eigenvalues.

(*c*) For what $c_1,\ c_2$ is your (c, A) observable? .... (6p)

Answer: c_1, c_2 such that $[(1 + c_1)^2 - c_2^2](4 + c_1 + 2c_2) \neq 0$ (no zero pole cancellation).

(*d*) Find $u(t) = Kx$ that makes $\mathcal{D} = \{x \in R^3 : cx = 0\}$ invariant, i.e., $cx(t) = 0, \forall t \geq 0$ if $cx(0) = 0$. .... (3p)

Answer: $y(t) \equiv 0$ implies that $\dot{y} \equiv 0$, from which we obtain $u(t)$.

(*e*) For what $c_1,\ c_2$ are the state trajectories $x(t)$ obtained in (d) bounded (you must prove your conclusion)? A solution $x(t)$ obtained in (d) means a solution to (1) when $u(t) = Kx$ is used and $cx(0) = 0$. .... (4p)

Answer: $c_1 \geq 0,\ c_2 \geq 0,\ c_1^2 + c_2^2 \neq 0$.

(*f*) Now suppose $c_1 = -2$. For what c_2 can we find a two dimensional minimal realization of the $r(s)$ such that for the minimal realization there exists $u(t) = Kx$ that makes both $cx(t) = 0, \forall t \geq 0$ if $cx(0) = 0$ and $\lim_{t\to\infty} x(t) = 0$. . (3p)

Answer: $c_2 = 1$.

3. (20p) Consider the transfer matrix

$$R(s) = \begin{bmatrix} \frac{\gamma}{(s+1)^2} & \frac{1}{(s+1)^2} \\ \frac{1}{s^2+s} & \frac{1}{s^2+s} \end{bmatrix},$$

where γ is a constant.

(a) Find the standard reachable realization. . . . . (6p)

(b) Compute the McMillan degree of $R(s)$. . . . . (6p)

(c) For the case $\gamma \neq 1$, find a minimal realization of $R(s)$ and verify your answer if you use Kalman decomposition. . . . . (8p)

Answer: Please see Solution to Exam Jan. 2015.

4. (20p) Consider the optimal control problem

$$\min_u J = \int_0^\infty (x^T Q x + u^2) dt \quad \text{s.t.} \quad \dot{x} = Ax + Bu, \ x(0) = x_0,$$

where,

$$A = \begin{bmatrix} k & 1 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad Q = q^2 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

(a) Suppose $q = 1$. Show that the associated algebraic Riccati equation (ARE) has a positive definite solution for any k. . . . . (7p)

Answer: We first get $c = [1 \ \ -1]$. Then the system is observable if $k \neq 0$. In this case we have $P > 0$. When $k = 0$ by solving the ARE we can see $P > 0$.

(b) Now suppose $q = 0$. What is the condition on k such that the ARE has a positive definite solution? . . . . (6p)

Answer: $-A$ must be a stable matrix, thus $k > 0$.

(c) Again suppose $q = 0$. Show that when the ARE has a positive definite solution, the closed-loop system has poles $\{-k, -1\}$. . . . . (4p)

Answer: From the ARE, we have $A - BB^T P = -P^{-1} A^T P$.

(d) When $q = 0$, what is the condition on k such that the optimal control exists? (3p)

Answer: $k \neq 0$.

5. (15p)

(a) This problem is related to the motion of a particle. Given a nonzero $b \in R^3$ (translational motion),

(1) show that for almost all skew symmetric matrices A ($A^T = -A$, rotation), (A, b) is controllable. . . . . (6p)

(2) give a geometric interpretation to those skew symmetric A such that (A, b) is not controllable. .... (2p)
Answer: Such an A matrix can always be determined by three parameters $a = (a_1, a_2, a_3)^T$. As long as b is not parallel or orthogonal to a, the pair will be controllable.

(b) Consider the discrete-time matrix Riccati equation associated with Kalman filters

$$\begin{aligned} P(t+1) &= AP(t)A^T - AP(t)C^T[CP(t)C^T + DRD^T]^{-1}CP(t)A^T + BQB^T \\ P(0) &= P_0. \end{aligned}$$

Assume that $A = a,\ C = 1,\ D = 1,\ R = r > 0,\ Q = 0$ and $P_0 = p_0 > 0$, where $|a| \neq 1$.

(1) Show that $\lim_{t\to\infty} P(t)$ exists. .... (4p)

(2) Denote $P = \lim_{t\to\infty} P(t)$, show that $|a - ak| < 1$, where $k = P(P+r)^{-1}$ is the Kalman gain. .... (3p)
Answer: Please see Solution to Exam Jan 2022.