1. If $a = 0$, then $V^* = \text{span}\{e_2, e_3, e_4\}$, and $V^* \cap \text{Im} B = \text{Im} B \neq 0$.
If $a \neq 0$, then the relative degree of the system is 2. Then the system is invertible. Therefore, the system is invertible iff $a \neq 0$.

(b) If $a \neq 0$, then the transmission zeros $s$ should satisfy the equation

$$s^2 + s + 2a = 0,$$

which gives $s_{1,2} = -\frac{1 \pm \sqrt{1 - 8a}}{2}$.
If $a = 0$, then $R^* = V^*$, and the number of transmission zeros are $\dim V^* - \dim R^* = 0$.
There is no transmission zero.

(c) High gain control is achievable when the zero dynamics is stable $\iff a > 0$.

2. In this problem, $A = \begin{pmatrix} 0 & 1 & 0 \\ -1 & -3 & -3 \end{pmatrix}$, $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $P = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\Gamma = \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix}$, $c = (-\alpha \ 0 \ 1)$ and $q = (1 \ 0)$.

(a) The solution to the Sylvester equation $A\Pi - \Pi\Gamma = -Pq$ is

$$\Pi = -\frac{1}{4} \begin{pmatrix} -1 & -1 \\ 1 & -1 \\ 1 & 1 \end{pmatrix}$$

(b) In order to have the whole system observable, we need 1) the pair $(c, A)$ to be observable, 2) the pair $(q, \Gamma)$ to be observable and 3) eigenvalues of $\Gamma$ are not the transmission zero of the system

$$\dot{x} = Ax + Pu$$

$$y = cx.$$
3) ⇒ \(\alpha \neq -1\)

Therefore, when \(\alpha \neq \pm 1\), the whole system is observable.

(c) The FIORP is solvable when the eigenvalues of \(\Gamma\) are not the transmission zero of the system

\[
\dot{x} = Ax + Bu
\]
\[
y = cx.
\]

The only case when this happens is when \(\alpha = -3\) and \(\beta = \sqrt{\Pi}\). When \(\alpha \neq -3\) or \(\beta \neq \sqrt{\Pi}\), the FIORP is solvable.

3.

The system could be rewrite as

\[
\begin{pmatrix}
\dot{\alpha_f} \\
\dot{\varphi} \\
\dot{r}
\end{pmatrix} =
\begin{pmatrix}
-2 & 0 & 0.76 \\
0 & 0 & 1 \\
-0.6 & -2.8 & 0
\end{pmatrix}
\begin{pmatrix}
\alpha_f \\
\varphi \\
r
\end{pmatrix}
+ \begin{pmatrix}
0 \\
0 \\
1
\end{pmatrix} u = A \begin{pmatrix}
\alpha_f \\
\varphi \\
r
\end{pmatrix} + bu,
\]

\[
\begin{pmatrix}
\dot{w}_1 \\
\dot{w}_2
\end{pmatrix} =
\begin{pmatrix}
0 & 2 \\
-2 & 0
\end{pmatrix}
\begin{pmatrix}
w_1 \\
w_2
\end{pmatrix} = \Gamma \begin{pmatrix}
w_1 \\
w_2
\end{pmatrix},
\]

\[
u = \begin{pmatrix}
1 \\
0
\end{pmatrix}
\begin{pmatrix}
w_1 \\
w_2
\end{pmatrix} = q \begin{pmatrix}
w_1 \\
w_2
\end{pmatrix}
\]

The OTIP is solved by \(\Pi\) and \(c\) that is the solution to the equations

\[
A\Pi - \Pi\Gamma = -bq
\]
\[
c\Pi = q,
\]

which gives the result

\[
\Pi = \begin{pmatrix}
-0.28 & -0.46 \\
-0.96 & -0.23 \\
0.46 & -1.95
\end{pmatrix}, \quad c = \begin{pmatrix}
0 & -0.97 & 0.11
\end{pmatrix}
\]