KTH Matematik

Solution to Final Exam of SF2842 Geometric Control Theory

December 19 2007, 14-19

We reserve the right to make corrections.

1. Determine if each of the following statements is *true* or *false* and **motivate** (no motivation no score) your answer briefly (for example, to show a statement is false, a counter-example is enough).

(a) Consider a linear system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{1}$$

where $x \in R^n$, $u \in R^m$, $y \in R^m$.
Let $S = ker(C)$. If (C, A) is observable, then $R^* = 0$ (the maximal reachability subspace in S). .... (2p)

Answer: false. A student should easily give a SISO system as a counter example.

(b) If system (1) has relative degree $(r_1, \cdots, r_m)$, then $dim(V^*) = n - \sum_{i=1}^{m} r_i$ and $R^* = 0$. .... (2p)

Answer: true. If one converts the system into the normal form, this can be seen easily

(c) Now suppose the system is influenced by disturbance Pw (namely we add Pw to the right hand side of equation (1)). Then the solvability of DDP is a necessary condition for solving any full information output regulation problem. .... (2p)

Answer: false. One can refer to an example in the lecture notes.

(d) Consider a nonlinear single-input system

$$\dot{x} = f(x) + g(x)u \tag{2}$$

where $x \in R^n$, $f, g \in C^\infty$ and $f(0) = 0$. Let $A = \frac{\partial f}{\partial x}|_{x=0}$, $b = g(0)$. If (A, b) is not controllable, then the system is not exactly linearizable around the origin. (2p)

Answer: true. Being "linear " controlllable is part of the requiement for exact linearization.

(e) We consider the same nonlinear system as in question (d). A necessary condition for the existence of an output mapping $h(x)$ such that the system has a relative degree $r \leq n$ at the origin is that the matrix A is not identically zero. ... (2p)

Answer: true. Otherwise the system can not have any relative degree.

2. Consider the system

$$\begin{aligned} \dot{x} &= \begin{pmatrix} -1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \\ 1 & 1 & 0 & -3 \end{pmatrix} x + \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} u \\ y &= (0 \ -1 \ 1 \ 0)x. \end{aligned}$$

(a) Find a feedback $u = Fx$ that would maximize the unobservable subspace. What is the unobservable subspace? .... (6p)

(b) Find R^* contained in V^*. .... (2p)

(c) Can we find a control $u = Fx$, where F is a friend of V^*, such that the closed-loop system is exponentially stable? .... (2p)

Answer: straight forward.

3. Consider the system

$$\begin{aligned} \dot{x} &= \begin{pmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \\ \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 2 & 3 & 4 \end{pmatrix} x. \end{aligned}$$

Choose two outputs out of the three (namely, disregard the third output), such that

(a) the noninteracting control problem is solvable, ... (6p)
and

(b) the zero dynamics is asymptotically stable. .... (4p)

Answer: take y_1 and y_3 for example.

4. Consider:

$$\begin{aligned} \dot{x} &= Ax + bu + Pw \\ \dot{w} &= \Gamma w \\ y &= cx, \end{aligned}$$

where

$$A = \begin{pmatrix} 0 & 9 & 0 \\ -1 & -\alpha & -1 \\ 0 & 0 & -\alpha \end{pmatrix}, \ \Gamma = \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix}, \ b = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}, \ P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}, \ c = \begin{pmatrix} 3 & 0 & 0 \end{pmatrix}$$

and α is a **positive constant**.

(a) Show the **existence** of an invariant subspace $x = \Pi(\alpha)w$ for the system when u is set to zero (no exact Π is needed) . . . . (2p)

Answer: we only need to show A is stable.

(b) Find the values of α, such that the reduced system

$$\begin{aligned} \dot{w} &= \Gamma w \\ y &= c\Pi(\alpha)w \end{aligned}$$

is not observable. . . . . (4p)

Answer: $\alpha = 1, 3$.

(c) Show for almost all values of α, the full information output regulation problem with the tracking error $e = y - w_2$ is solvable (*a solution is not required*) . What are the values of α that may make the problem unsolvable?
. . . . (4p)

Answer: $\alpha = 1, 3$, but for different reasons.

5. Consider in a neighborhood N of the origin

$$\begin{aligned} \dot{x}_1 &= \alpha \tan(x_1) + x_2 \tan(x_1) + 2x_3 + \cos(x_3)u \\ \dot{x}_2 &= -\alpha \sin^2(x_1) - \sin^3(x_1) - x_2 + x_3^3 \\ \dot{x}_3 &= x_3 - \cos(x_1)u \\ y &= x_3, \end{aligned}$$

where α is a constant.

(a) Convert the system into the normal form (*you need to specify the new coordinates explicitly, but do not need to calculate the right hand side of the normal form in every detail*). . . . . (4p)

Answer: Straight forward.

(b) Analyze the stability of the zero dynamics with respect to the value of α. (4p)

Answer: Straight forward.

(c) Consider the same nonlinear system but without the output. Show that the exact linearization problem is not solvable (*Hint: this does not have to involve a lot of calculations*). . . . . (2p)

Answer: since the linearized system is not controllable.