Geometric Control Theory[1]

Lecture notes by

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[1]This work is partially based on the earlier lecture notes by Lindquist, Mari and Sand.

Contents

Preface

Various versions of the notes have been used for the course "Geometric Control Theory" given at the Royal Institute of Technology (KTH), and its predecessor "Advanced Systems Theory". The first version was written by Anders Lindquist and Janne Sand, and was later revised and extended by Jorge Mari. A major revision and addition was done by Xiaoming Hu in 2002.

I would like to express my gratitude to Dr Ryozo Nagamune and Docent Ulf Jönsson for their careful reading and constructive comments on the 2002 version of the notes.

Some minor changes and updates have been made every year since 2002. In 2006, some new material was added and the title was changed to "Geometric Control Theory"; In 2011 Chapter 9 was rewritten.

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Notations and definitions

For easy reference we recall here some notations and definitions. Assume $\mathcal{X}$ and $\mathcal{Y}$ are finite dimensional vector spaces over the real field. Let n be the dimension of $\mathcal{X}$, which is then isomorphic to $\mathbb{R}^n$.

(1) *Image space*. Given a map $A : \mathcal{X} \to \mathcal{Y}$, the *image space* of A is defined as

$$\text{Im } A := \{y \in \mathcal{Y} \, : \, y = Ax, \text{ for some } x \in \mathcal{X}\}.$$

It is a subspace of $\mathcal{Y}$. We shall often use the same symbol for the map A as for its matrix representation once bases in $\mathcal{X}$ and $\mathcal{Y}$ have been chosen.

(2) *Linear span*. Given a vector space $\mathcal{V}$ over the field $\mathcal{R}$, let $v_1, \cdots, v_m \in \mathcal{V}$. The span of these vectors is

$$span\{v_1, \cdots, v_m\} = \{\alpha_1 v_1 + \cdots + \alpha_m v_m | \alpha_1, \cdots, \alpha_m \in \mathcal{R}\}.$$

(3) *Null space*. Given a map $A : \mathcal{X} \to \mathcal{Y}$, the *null space* , or *kernel*, of A is defined as

$$\ker A := \{x \in \mathcal{X} : Ax = 0\}.$$

It is a subspace of $\mathcal{X}$.

(4) *Preimage*. Let $\mathcal{W}$ be any set of $\mathcal{Y}$. The *pre-image* of $\mathcal{W}$ under the map A is

$$A^{-1}\mathcal{W} := \{x \in \mathcal{X} : Ax \in \mathcal{W}\}.$$

Observe that A need not be invertible, so beware of the distinction between preimage and inverse.

(5) *A-invariant subspace*. A subspace $\mathcal{V}$ of $\mathcal{X}$ is *A-invariant* if $A\mathcal{V} \subseteq \mathcal{V}$.

(6) *Reachable subspace*. Given the pair of conformable matrices $A_{n\times n}$ and $B_{n\times k}$, the *reachable subspace* of (A, B), denoted by $\langle A | \text{Im } B\rangle$, is defined as $\langle A | \text{Im } B\rangle := \text{Im } \Gamma$, where Γ is the reachability matrix $[B \, AB \, \ldots \, A^{n-1}B]$. This is an $n \times nk$ matrix. The reachable subspace is A-invariant.

(7) $\langle A | \text{Im } E\rangle$. The minimal A-invariant subspace that contains the subspace $\text{Im } E$ is denoted by $\langle A | \text{Im } E\rangle$.

(8) *Reachability subspace.* Given a matrix pair (A, B), a subspace $\mathcal{R}$ is called a *reachability subspace* if there are matrices F and G such that

$$\mathcal{R} = \langle A + BF | \operatorname{Im} BG \rangle.$$

(9) *Unobservable subspace.* Given the matrix pair (C, A), the *unobservable subspace* is defined as $\ker \Omega$, where Ω is the observability matrix

$$\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

The unobservable subspace is A-invariant.

(10) The map A is *injective* if $Ax_1 = Ax_2$ implies $x_1 = x_2$.

(11) If $A : \mathcal{X} \to \mathcal{Y}$ and $\mathcal{V} \subseteq \mathcal{X}$, the *restriction* of A to $\mathcal{V}$ is denoted by $A|_{\mathcal{V}}$.

(12) Given the two finite-dimensional subspaces $\mathcal{V}$ and $\mathcal{W}$, the *vector sum* is defined as

$$\mathcal{V} + \mathcal{W} := \{v + w : v \in \mathcal{V},\ w \in \mathcal{W}\}.$$

If $\mathcal{V}$ and $\mathcal{W}$ are linearly independent we write the sum as $\mathcal{V} \oplus \mathcal{W}$. Note that in these notes $\oplus$ does *not* denote orthogonal vector sum!

(13) *Hypersurface.* Suppose N is an open set in R^n. The set M is defined as

$$M = \{x \in N : \lambda_i(x) = 0, i = 1, \ldots, n - m\}$$

where λ_i are smooth functions.

If $\operatorname{rank} \begin{bmatrix} \frac{\partial \lambda_1}{\partial x} \\ \vdots \\ \frac{\partial \lambda_{n-m}}{\partial x} \end{bmatrix} = n - m \ \forall x \in M$, then M is a (hyper)surface of dimension m.

(14) *Lie derivative.* In local coordinates, Lie derivative is represented by

$$L_f \lambda := \sum_{i=1}^{n} \frac{\partial \lambda}{\partial x_i} f_i,$$

where f is a vector and λ is a scalar function.

(15) *Lie bracket.* Lie bracket of the two vector fields f and g is defined according to the rule:

$$[f, g](\lambda) := L_f L_g \lambda - L_g L_f \lambda.$$

In local coordinates the expression of $[f, g]$ is given as

$$\frac{\partial g}{\partial x} f - \frac{\partial f}{\partial x} g.$$