1. Consider the system

\[ \dot{x} = Ax + Bu = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} x + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} u \]

\[ y = Cx = (1 0 0 0)x. \]

(a) Is the system controllable? ...................................................(1p)

(b) Let \( x(t, u) \) denote the solution to (1) with control \( u(t) \) and initial condition \( x(0, u) = x_0 \). Compute the subspace \( S \) of initial conditions \( x_0 \) that make \( x(t, u) \in \text{Ker} C \ \forall t \geq 0 \) for some \( u(t) \), and design such a \( u(t) \) as feedback control. .........................................................(3p)

(c) For any \( x_0 \in S \), where \( S \) is the subspace you computed in (b), and any \( t_1 > 0 \), can we always find a \( u(t) \) such that \( x(t_1, u) = 0? \) .......................(1p)

(d) For any \( x_0 \in S \), where \( S \) is the subspace you computed in (b), and any \( t_1 > 0 \), can we always find a \( u(t) \) such that \( x(t_1, u) = 0 \), and \( x(t, u) \in S, \ 0 \leq t \leq t_1? \) (2p)

2. Consider an observable system

\[ \dot{x} = Ax + Bu \]
\[ y = Cx, \]

where \( x \in R^n, \ u \in R^1, \ y \in R^1. \)

(a) Show the controllable subspace \( \mathcal{R} \) is \((A + BF)\)-invariant for any \( F \).............(1p)

(b) List all controllability subspaces. ..............................................(1p)

(c) Show that \((C, A + BF)\) is also observable for almost all \( F \), namely the elements of those \( F \) that make \((C, A + BF)\) not observable can be defined by a set of algebraic constraints.........................(3p)
3. Consider

\[
\begin{align*}
\dot{x} &= Ax + Bu + Ew \\
y &=Cx,
\end{align*}
\]

where

\[
A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & a & 0 \\ 2 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad C = (1 \ 0 \ 0), \quad E = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix},
\]

where \(a\) and \(d_1, d_2, d_3\) are constants.

(a) For what \(E\) is \(DDP\) solvable? ......................................... (2p)

(b) For what \(a\) can we find a \(u = Fx\) that solves the \(DDP\) problem while makes the closed-loop system stable? i.e. \(A + BF\) has only eigenvalues with negative real part............................................................ (2p)

(c) What is \(R^*\)? ................................................................. (1p)

4. Consider

\[
\begin{align*}
\dot{x}_1 &= x_1 + x_3 + u_1 \\
\dot{x}_2 &= -x_1 + x_3 - u_1 \\
\dot{x}_3 &= x_2 - x_3 + x_4 + u_2 \\
\dot{x}_4 &= 2x_1 + x_4 + u_1 \\
y_1 &= x_1 + x_2 \\
y_2 &= x_4
\end{align*}
\]

(a) What is the relative degree for the system? ......................... (1p)

(b) Convert the system into the normal form and compute the zero dynamics.(3p)

(c) When \(y(t) = 0 \forall t \geq 0\), what happens to \(x(t)\) as \(t \to \infty\)? ............... (1p)