Homework 2.

1\. a). $C_1 B = (1\ 0\ 0\ 0)\begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} = (0\ \ 0).$

$$C_1 A B = (1\ 0\ 0\ 0)\begin{pmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix}$$

$$= (0\ 1\ 1\ 0)\begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} = (1\ \ 1) \neq (0\ \ 0).$$

$\Rightarrow r_1 = 2.$

$$C_2 B = (0\ a\ 1\ 1)\begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} = (2\ \ a+1) \neq (0\ \ 0).$$

$\Rightarrow r_2 = 1.$

$L = \begin{pmatrix} 1 & 1 \\ 2 & a+1 \end{pmatrix}$ nonsingular when the system has a relative degree. $\Rightarrow a \neq 1.$

b). $\xi_1^1 = x_1 = 0.$

$\dot{\xi}_1^1 = \xi_2^1 = x_2 + x_3 = 0.$

$\dot{\xi}_2^1 = \dot{x}_2 + \dot{x}_3 = x_4 + u_1 + u_2 = 0.$

$\xi_1^2 = a x_2 + x_3 + x_4 = 0$

$\dot{\xi}_1^2 = (1-a)x_1 + x_2 + x_3 + x_4 + 2u_1 + (a+1)u_2 = 0.$

$z = x_2 + x_3 - x_4.$

$\dot{z} = -x_2 - x_3 + x_4 = -z.$

c). $P_\Sigma(s_0) = \det\begin{pmatrix} sI - A & B \\ -C & 0 \end{pmatrix}$

$$= \begin{vmatrix} s & -1 & -1 & 0 & 0 & 0 \\ 1 & s & 0 & -1 & 0 & 1 \\ -1 & 0 & s & 0 & 1 & 0 \\ 0 & -1 & -1 & s & 1 & 1 \\ \cancel{-1} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \\ 0 & -a & -1 & -1 & 0 & 0 \end{vmatrix} = \begin{vmatrix} -1 & -1 & 0 & 0 & 0 \\ s & 0 & -1 & 0 & 1 \\ 0 & s & 0 & 1 & 0 \\ -1 & -1 & s & 1 & 1 \\ -a & -1 & -1 & 0 & 0 \end{vmatrix}.$$

$$= -\begin{vmatrix} 0 & -1 & 0 & 1 \\ s & 0 & 1 & 0 \\ -1 & s & 1 & 1 \\ -1 & -1 & 0 & 0 \end{vmatrix} + \begin{vmatrix} s & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ -1 & s & 1 & 1 \\ -a & -1 & 0 & 0 \end{vmatrix}.$$

$$= -\begin{vmatrix} s & 1 & 0 \\ -1 & 1 & 1 \\ -1 & 0 & 0 \end{vmatrix} + \begin{vmatrix} s & 0 & 1 \\ -1 & s & 1 \\ -1 & -1 & 0 \end{vmatrix} - \begin{vmatrix} s & -1 & 1 \\ -1 & s & 1 \\ -a & -1 & 0 \end{vmatrix}.$$

$$= +1 + (1+s+s) - (a+1+as+s).$$

$$= +1+1+2s-a-1-as-s = (1-a)s+1-a.$$

$$\cancel{= (-3-a)s-3-a=0.}$$

$\Rightarrow s=-1$ when $a \neq 1$.

2. a). sylvester equation.

$$A\Pi - \Pi\Gamma = -bq. \quad \Pi \in \mathbb{R}^{3\times 2}$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{pmatrix}\begin{pmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{21} & \Pi_{22} \\ \Pi_{31} & \Pi_{32} \end{pmatrix} - \begin{pmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{21} & \Pi_{22} \\ \Pi_{31} & \Pi_{32} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}(1 \quad 0).$$

$$\begin{pmatrix} \Pi_{21} & \Pi_{22} \\ \Pi_{31} & \Pi_{32} \\ -\Pi_{11}-3\Pi_{21}-3\Pi_{31} & -\Pi_{12}-3\Pi_{22}-3\Pi_{32} \end{pmatrix} - \begin{pmatrix} -\Pi_{12} & \Pi_{11} \\ -\Pi_{22} & \Pi_{21} \\ -\Pi_{32} & \Pi_{31} \end{pmatrix} = -\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

$$\begin{cases} \Pi_{21}+\Pi_{12}=0 \Rightarrow \Pi_{21}=-\Pi_{12} \\ \Pi_{22}-\Pi_{11}=0 \Rightarrow \Pi_{22}=\Pi_{11} \\ \Pi_{31}+\Pi_{22}=0 \Rightarrow \Pi_{31}=-\Pi_{22}=-\Pi_{11} \\ \Pi_{32}-\Pi_{21}=0 \Rightarrow \Pi_{32}=\Pi_{21}=-\Pi_{12}. \\ -\Pi_{11}-3\Pi_{21}-3\Pi_{31}+\Pi_{32}=-1 \\ -\Pi_{12}-3\Pi_{22}-3\Pi_{32}-\Pi_{31}=0. \end{cases}$$

$$-\Pi_{11}-3(-\Pi_{12})-3(-\Pi_{11})+(-\Pi_{12}) = 2\Pi_{11}+2\Pi_{12}=-1.$$

$$-\Pi_{12}-3\Pi_{11}-3(-\Pi_{12})-(-\Pi_{11}) = -2\Pi_{11}+2\Pi_{12}=0. \Rightarrow \Pi_{11}=\Pi_{12}$$

$$\Rightarrow \Pi_{11} = -\frac{1}{4}.$$

$$\Rightarrow \Pi = \frac{1}{4}\begin{pmatrix} -1 & -1 \\ 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

b) we want (C, A) to be observable and no eigenvalue of T is transmission zero of the system.

$$\Omega = \begin{pmatrix} C \\ CA \\ CA^2 \end{pmatrix} = \begin{pmatrix} c_1 & c_2 & 1 \\ -1 & c_1-3 & c_2-3 \\ 3-c_2 & 8-3c_2 & 6-3c_2+c_1 \end{pmatrix}.$$

$$\det \Omega = c_1(c_1-3)(6-3c_2+c_1) + c_2(c_2-3)(3-c_2) - (8-3c_2)$$
$$-(c_1-3)(3-c_2) + c_2(6-3c_2+c_1) - c_1(c_2-3)(8-3c_2)$$
$$= c_1^3 - 3c_1^2c_2 + 3c_1c_2^2 - c_2^3 + 3c_1^2 - 6c_1c_2 + 3c_2^2$$
$$+3c_1 - 3c_2 + 1$$
$$= (c_1 - c_2 + 1)^3 \neq 0 \quad \text{since } c_1 - c_2 + 1 \neq 0.$$

Eigenvalues of T is $\pm i$.

Transmission zero:

$$P_\Sigma(s) = \det \begin{pmatrix} sI-A & B \\ -C & 0 \end{pmatrix}$$

$$= \begin{vmatrix} s & -1 & 0 & 0 \\ 0 & s & -1 & 0 \\ 1 & 3 & s+3 & 1 \\ -c_1 & -c_2 & -1 & 0 \end{vmatrix} = s^2 + c_2 s + c_1$$

For $s = \pm i$, $P_\Sigma(s)$ should not be zero

$\Rightarrow -1 \pm i c_2 + c_1 \neq 0$.

Consider real c_1, c_2, the only possibility for this equal to zero is $c_1 = 1$, $c_2 = 0$.

c). $C\Pi = q \Rightarrow \begin{cases} -c_1 + c_2 + 1 = 4 \\ -c_1 - c_2 + 1 = 0 \end{cases} \Rightarrow c_1 = -1, \; c_2 = 2.$

3. a). $S = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ has eigenvalues $0, \pm i$.

Rosenbrock matrix: $\begin{pmatrix} sI-A & B \\ -C & 0 \end{pmatrix}$.

$$P_\Sigma(s) = \det \begin{pmatrix} sI-A & B \\ -C & 0 \end{pmatrix}$$

$$= \begin{vmatrix} s & -1 & 0 & -1 & 0 & 0 \\ 0 & s-1 & 0 & 0 & 1 & 0 \\ 0 & 0 & s+2 & 0 & 0 & 1 \\ -1 & 0 & \alpha & s+1 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \end{vmatrix}$$

$= -s + (\alpha - 2)$. cannot have zero as root.

$\Rightarrow \alpha \neq 2$ (consider only real α).

We also need (A, B) to be controllable, but it is always true since

$$(B \;\; AB \;\; A^2B) = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 & -\alpha-1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -2 & 0 & 4 \\ 0 & 1 & 0 & -\alpha-1 & 1 & 3\alpha+2 \end{pmatrix}$$

already has rank 4.

b). Besides the conditions in a), we also need $\left((C \;\; -Q), \begin{pmatrix} A & P \\ 0 & S \end{pmatrix} \right)$ detectable.

In fact, when $\alpha = 0$, the pair (C, A) is not observable. So $\left((C \;\; -Q), \begin{pmatrix} A & P \\ 0 & S \end{pmatrix} \right)$ is not observable.

(also not detectable by further argument).

For the other cases, the pair $\left((C\ -Q), \begin{pmatrix} A & P \\ 0 & S \end{pmatrix}\right)$

is always observable.

So the EFORP is solvable when $\alpha \neq 0$ and $\alpha \neq 2$.

c). $\alpha = 1$.

$$A = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 1 & 0 & -1 & 0 \end{pmatrix}$$

We want to solve Sylvester equation

$$\Pi S = A\Pi + P + B\Gamma \qquad ①$$

$$0 = C\Pi - Q \qquad ②$$

$$② \Rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \Pi_{11} & \Pi_{12} & \Pi_{13} \\ \Pi_{21} & \Pi_{22} & \Pi_{23} \\ \Pi_{31} & \Pi_{32} & \Pi_{33} \\ \Pi_{41} & \Pi_{42} & \Pi_{43} \end{pmatrix} - \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix} = 0$$

$$\Rightarrow \begin{array}{l} \Pi_{11} = 2, \quad \Pi_{12} = 0, \quad \Pi_{13} = 0, \\ \Pi_{41} = 0, \quad \Pi_{42} = 3, \quad \Pi_{43} = 0. \end{array}$$

$$① \Rightarrow \begin{pmatrix} 2 & 0 & 0 \\ \Pi_{21} & \Pi_{22} & \Pi_{23} \\ \Pi_{31} & \Pi_{32} & \Pi_{33} \\ 0 & 3 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 1 & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ \Pi_{21} & \Pi_{22} & \Pi_{23} \\ \Pi_{31} & \Pi_{32} & \Pi_{33} \\ 0 & 3 & 0 \end{pmatrix} +$$

$$+ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \Gamma_{11} & \Gamma_{12} & \Gamma_{13} \\ \Gamma_{21} & \Gamma_{22} & \Gamma_{23} \end{pmatrix}$$

row 1: $(0 \;\; 2 \;\; 0) = (\Pi_{21} \;\; \Pi_{22}+3 \;\; \Pi_{23}) + (0 \;\; 0 \;\; 0) + (0 \;\; 0 \;\; 0)$

$\Rightarrow (\Pi_{21} \;\; \Pi_{22} \;\; \Pi_{23}) = (0 \;\; -1 \;\; 0)$

row 2: $(1 \;\; 0 \;\; 0) = (0 \;\; -1 \;\; 0) + (0 \;\; 0 \;\; 0) + (\Gamma_{11} \;\; \Gamma_{12} \;\; \Gamma_{13})$

$\Rightarrow (\Gamma_{11} \;\; \Gamma_{12} \;\; \Gamma_{13}) = (1 \;\; 1 \;\; 0)$

row 3: $(-3 \;\; 0 \;\; 0) = (2-\Pi_{31} \;\; -\Pi_{32}-3 \;\; -\Pi_{33}) + (\Gamma_{21} \;\; \Gamma_{22} \;\; \Gamma_{23})$

row 4: $(-\Pi_{32} \;\; \Pi_{31} \;\; 0) = (-2\Pi_{31} \;\; -2\Pi_{32} \;\; -2\Pi_{33}) + (0 \;\; 0 \;\; 1) + (\Gamma_{21} \;\; \Gamma_{22} \;\; \Gamma_{23})$

row 3 - row 4 $\Rightarrow (\Pi_{32}-3 \;\; -\Pi_{31} \;\; 0) = (2+\Pi_{31} \;\; \Pi_{32}-3 \;\; \Pi_{33}) + (0 \;\; 0 \;\; -1)$

$\Rightarrow \Pi_{31} = -1 \quad \Pi_{32} = 4, \quad \Pi_{33} = 1$

$(\Gamma_{21} \;\; \Gamma_{22} \;\; \Gamma_{23}) = (-3 \;\; 0 \;\; 0) + (\Pi_{31}-2 \;\; \Pi_{32}+3 \;\; \Pi_{33}) = (-6 \;\; 7 \;\; 1)$

$$\Rightarrow \Pi = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ -1 & 4 & 1 \\ 0 & 3 & 0 \end{pmatrix}, \quad \Gamma = \begin{pmatrix} 1 & 1 & 0 \\ -6 & 7 & 1 \end{pmatrix}.$$

By using Matlab, we can find K matrix to stabilize $A+BK$. For example $K = \begin{pmatrix} 1 & 3 & 0 & 1 \\ 1 & 1/2 & -1 & 1/2 \end{pmatrix}$.

$u = K(x - \Pi w) + \Gamma w$ is the answer.