Selected Problems in Optimal Control

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Contents

1. Discrete Time Optimal Control

1.1 An investor receives an annual income of amount x_k (each year k). From the x_k received, the investor may reinvest one part $x_k - u_k$ and keep u_k for spending. The reinvestment results in an increase of the capital income as

$$x_{k+1} = x_k + \theta(x_k - u_k)$$

where $\theta > 0$ is given.

The investor wants to maximize his total consumption over N years, i.e. he wants to maximize the utility $\sum_{k=0}^{N-1} u_k$. The resulting optimization problem is

$$\max \sum_{K=0}^{N-1} u_k \quad \text{subj. to} \quad \begin{cases} x_{k+1} = x_k + \theta(x_k - u_k) \\ 0 \le u_k \le x_k, \ x_0 > 0 \text{ is given} \end{cases}$$

(a) Formulate the dynamic programming recursion that solves this optimization problem.

(b) Solve the problem when $N = 3$, $x_0 = 1$, $\theta = 0.1$.

1.2 Consider the discrete optimal control problem

$$\min \sum_{k=0}^{1} (|x_k| + 5|u_k|) \quad \text{s.t} \quad \begin{cases} x_{k+1} = 0.5x_k + u_k \\ x_k \in X_k; u_k \in \{-1, -0.5, 0, 0.5, 1\} \end{cases}$$

where the state space is defined by

$$X_0 = \{-2\}, \ X_1 = \{-2, -1, 0, 1, 2\}, \ X_2 = \{0\}$$

Solve the problem using dynamic programming.
Hint: It may be useful to introduce the control constraint sets $U(k, x)$ that specify the feasible control values for each $x_k \in X_k$.

1.3 In this problem you will solve the following optimal control problem

$$\min \sum_{k=0}^{2} (x_k^2 + u_k^2) \quad \text{subj. to} \quad \begin{cases} x_{k+1} = x_k + u_k, \quad x_0 = 0; \ x_3 = 2 \\ u_k \in U_k(x_k) = \{u : 0 \le x_k + u \le 2; u \text{ is an integer}\} \end{cases}$$

(a) Formulate the dynamic programming algorithm for this problem.

(b) Use the dynamic programming recursion to find the optimal solution.

1.4 Let z_k denote the number of university teachers at time k and let y_k denote the number of scientists at time k. The number of teachers and scientists evolve according to the equations

$$\begin{aligned} z_{k+1} &= (1 - \delta)z_k + \gamma z_k u_k, \\ y_{k+1} &= (1 - \delta)y_k + \gamma z_k(1 - u_k) \end{aligned}$$

where $0 < \delta < 1$ and $\gamma > 0$ are constants and $0 < \alpha \le u_k \le \beta < 1$. The interpretation of these equations is that by controlling the funding of the university system it is possible to control the fraction of newly educated teachers that become scientists, i.e. funding affects the control u_k. We assume $z_0 > 0$ and $y_0 = 0$ and in the above equations we allow both z_k and y_k to be non-integer valued in order to simplify the problem. This is a reasonable approximation if, for example, one unit is 10^5 persons.

(a) We would like to determine the control u_k so that the number of scientists is maximal at year N. Formulate the dynamic programming recursion that solves this problem.

(b) Use dynamic programming to solve the problem in (a) when $\delta = 0.5$, $\gamma = 0.5$, $\alpha = 0.2$, $\beta = 0.8$, $z_0 = 1$, $y_0 = 0$ and $N = 2$.

2. Infinite Horizon Discrete Time Optimal Control

2.1 Consider the optimal control problem

$$\min \sum_{k=0}^{\infty} (x_k^2 + u_k^2) \quad \text{subj.to} \quad x_{k+1} = 2x_k + u_k, \; x_0 = 2$$

(a) Use the Bellman equation

$$V(x) = \min_u \{f_0(x,u) + V(f(x,u))\}$$

to compute the optimal control and the optimal cost.

(b) Compute the eigenvalue of the closed loop system.

3. Model Predictive Control

3.1 The model predictive control algorithm

(i) Measure $x_{t|t} := x_t$.

(ii) Determine $u_{t|t}$ by solving

$$\begin{aligned} &\min x_{t+2|t}^2 + u_{t+1|t}^2 + u_{t|t}^2 \\ &\text{subj. to} \quad \begin{cases} x_{t+k+1|t} = x_{t+k|t} + u_{t+k|t}, \; k = 0, 1 \end{cases} \end{aligned}$$

(iii) Apply $u_t := u_{t|t}^*$

(iv) Let $t := t + 1$ and go to 1.

can be solved either using on-line optimization or by obtaining an explicit solution.

(a) Determine an explicit solution $u_{t|t} = \mu(x_{t|t})$ to this MPC problem.

(b) Is the closed loop system stable? In other words is the system $x_{t+1} = x_t + \mu(x_t)$ converging to zero, where $\mu(\cdot)$ is the feedback law computed in problem (a).?

3.2 In this problem we investigate two model predictive control problems.

(a) The model predictive control algorithm

(i) Measure $x_{t|t} := x_t$.

(ii) Determine $u_{t|t}$ by solving

$$\min 2|x_{t+1|t}| + |u_{t|t}|$$
$$\text{subj. to} \quad \begin{cases} x_{t+1|t} = x_{t|t} + u_{t|t}, \\ |u_{t|t}| \le 1, \end{cases}$$

(iii) Apply $u_t := u^*_{t|t}$

(iv) Let $t := t + 1$ and go to i.

can be solved either using on-line optimization or by obtaining an explicit solution. Determine an explicit solution $u_{t|t} = \mu(x_{t|t})$ to this MPC problem.

(b) Solve the same problem when step (ii) is replaced by

$$\min 2|x_{t+2|t}| + |u_{t+1|t}| + |u_{t|t}|$$
$$\text{subj. to} \quad \begin{cases} x_{t+k+1|t} = x_{t+k|t} + u_{t+k|t}, \ k = 0, 1 \\ |u_{t+k|t}| \le 1, \ k = 0, 1 \end{cases}$$

4. Dynamic Programming in Continuous Time

4.1 This problem consists of two questions.

(a) Determine which of the following partial differential equations (a) – (d) corresponds to the following optimal control problem

$$\min_u x(1)^2 \quad \text{s.t} \quad \begin{cases} \dot{x} = 2u, \quad x(0) = x_0 \\ |u| \le 1 \end{cases}$$

(a) $-V_t = -2V_x \text{sign}(V_x)$, $V(1, x) = x^2$

(b) $-V_t = -2V_x \text{sign}(V_x)$, $V(1, x) = 2x$

(c) $-V_t = -2V_x$, $V(1, x) = x^2$

(d) $-V_t = -\sin(V_x)(1 + 2V_x)$, $V(1, x) = x^2$

(b) Consider the optimal control problem

$$J(x_0) = \min_u \int_0^1 f_{01}(x, u)dt + \int_1^2 f_{02}(x, u)dt$$
$$\text{s.t.} \quad \begin{cases} \dot{x} = f_1(x, u), \quad 0 \le t \le 1, \quad x(0) = x_0 \\ \dot{x} = f_2(x, u), \quad 1 \le t \le 2 \end{cases}$$

Below are two attempts of solving the problem. They cannot both be correct (and may both be wrong). Find and explain the error(s) in the reasoning. Is any of the two attempts correct?

Attempt 1:

$$\begin{aligned} J^*(x_0) = \ & \min_{u_1} \int_0^1 f_{01}(x_1, u_1)dt \quad && + \ \min_{x_1(1)} \min_u \int_1^2 f_{02}(x_2, u_2)dt \\ & s.t. \ \dot{x}_1 = f_1(x_1, u_1), \ x_1(0) = x_0 && \quad s.t. \ \dot{x}_2 = f_2(x_2, u_2), \ x_2(1) = x_1(1) \\ = & \min_{x_1(1)} J_1^*(x_0) + J_2^*(x_1(1)) \end{aligned}$$

Attempt 2:

$$\begin{aligned} J^*(x_0) = & \min_u\{\int_0^1 f_{01}(x,u)dt && + \min_u \int_1^2 f_{02}(x,u)dt \ \} \\ & s.t.\ \dot{x} = f_1(x,u),\ x(0) = x_0 && s.t.\ \dot{x} = f_2(x,u), \\ \\ = & \min_u \left\{ \int_0^1 f_{01}(x,u)dt + J_2^*(x(1)) \right\} \\ & s.t.\ \dot{x} = f_1(x,u),\ x(0) = x_0 \end{aligned}$$

where

$$\begin{aligned} J_k^*(x_0) = & \min_u \int_{k-1}^k f_{0k}(x_k,u)dt \\ & s.t.\ \dot{x}_k = f_k(x_k,u),\ x_k(k-1) = x_0 \end{aligned}$$

4.2 Consider the following value function (cost-to-go function)

$$V(t_0,x_0) = \max_{u\in\mathbf{R}} \int_{t_0}^{T} \sqrt{u(s)}ds \quad \text{s.t.} \quad \dot{x}(t) = \beta x(t) - u(t),\ x(t_0) = x_0$$

where $\beta > 0$. Verify that $V(t,x) = f(t)\sqrt{x}$, where

$$f(t) = \sqrt{\frac{e^{\beta(T-t)} - 1}{\beta}}$$

Comment: Note that the value function is only defined on $[0,T] \times \boldsymbol{R}_+$, where $\boldsymbol{R}_+ = (0,\infty)$ (i.e. $V : [0,T] \times \boldsymbol{R}_+ \to \boldsymbol{R}_+$). The theorems presented in the course are also valid when the domain is restricted to such a set.

4.3 Consider the nonlinear optimal control problem

$$\min x(1)^2 + \int_0^1 (x(t)u(t))^2 dt \quad \text{subj. to} \quad \dot{x}(t) = x(t)u(t), \quad x(0) = 1$$

Solve the problem using dynamic programming.

Hint: Use the ansatz $V(t,x) = p(t)x^2$.

4.4 In this problem you will investigate an optimal control problem with autonomous dynamics. The constraint that the final state state is zero has certain implications that you will investigate.

(*a*) Consider the optimal control problem

$$\min \int_0^{t_f} (x^2 + u^2)dt \quad \text{s.t.} \quad \begin{cases} \dot{x} = x + u,\ x(0) = x_0 \\ x(t_f) = 0,\ t_f > 0 \end{cases}$$

where the final time is a free variable to be optimized. If we apply dynamic programming then we get two possible feedback solutions

$$u = -(1 \pm \sqrt{2})x$$

Are both optimal?

(*b*) Now consider the more general case

$$\min \int_0^{t_f} (\|x\|^2 + u^2)dt \quad \text{s.t.} \quad \begin{cases} \dot{x} = Ax + Bu, \ x(0) = x_0 \\ x(t_f) = 0, \ t_f > 0 \end{cases}$$

where (A, B) is a controllable pair. How do you compute the optimal solution? Justify your answer.

(*c*) Solve (*b*) when

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad x_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

4.5 The purpose of this problem is investigate continuous time dynamic programming applied to optimal control problems with discounted cost and apply it to an investment problem.

(*a*) Consider the optimal control problem

$$\begin{aligned} &\min_u e^{-\alpha T}\Phi(x(T)) + \int_0^T e^{-\alpha t} f_0(t, x(t), u(t))dt \\ &\text{subj.to.} \quad \dot{x}(t) = f(t, x(t), u(t)), \quad x(t_0) = x_0 \end{aligned}$$

Let $V(x, t) = e^{-\alpha t}W(x, t)$. Use the Hamilton-Jacobi-Bellman Equation (HJBE) to derive a new (HJBE) for $W(t, x)$ (the discounted cost HJBE).

(*b*) Solve the following investment problem

$$\max_u \int_0^T e^{-\alpha t}\sqrt{u(t)}dt \quad \text{subj. to} \quad \dot{x}(t) = \beta x(t) - u(t), \ x(0) = x_0$$

where x_0 is the initial amount of savings, u is the rate of expenditure, and β is the interest rate on the savings.
Hint: You may use problem (a) with $W(t, x) = w(t)\sqrt{x}$, where $w(t)$ is a function to be determined.

5. Pontryagin Minimum Principle

5.1 Consider the optimal control problem.

$$\min \int_0^1 (3x(t)^2 + u(t)^2)dt \quad \text{subj. to} \quad \dot{x}(t) = x(t) + u(t), \quad x(0) = x_0$$

(*i*) Determine the optimal feedback control.

(*ii*) Determine the optimal cost.

5.2 We will solve two similar optimal control problems.

(*a*) Use PMP to solve

$$\min \int_0^2 (u_1(t)^2 + u_2(t)^2)dt \quad \text{subj. to} \quad \begin{cases} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}, \\ x(0) = 0, \ x(2) \in S_2 \end{cases}$$

where $S_2 = \{x \in \mathbf{R}^2 : x_2^2 - x_1 + 1 = 0\}$.

(b) Use PMP to solve

$$\min \int_0^2 (u_1(t)^2 + u_2(t)^2)dt \quad \text{subj. to} \quad \begin{cases} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}, \\ x(0) \in S_0, \ x(2) \in S_2 \end{cases}$$

where $S_0 = \{x \in \mathbf{R}^2 : x_2^2 + x_1 = 0\}$ and S_2 is as above.

5.3 Use PMP to solve the optimal control problem

$$\min \int_0^1 4(2-u)x dt \quad \text{subject to} \quad \begin{cases} \dot{x} = 2(2u-1)x, \quad x(0) = 2, \ x(1) = 4 \\ 0 \le u \le 2 \end{cases}$$

Hint: First prove that $x(t)$ has constant sign on $t \in [0,1]$.

5.4 Consider the optimal control problem

$$\min \int_0^{t_f} u(t)dt \quad \text{s.t.} \quad \begin{cases} \dot{x}(t) = -x(t) + u(t), \quad x(0) = x_0 \quad x(t_f) = 0 \\ u \in [0, m] \end{cases} \tag{5.1}$$

(a) Suppose t_f is fixed. For what values of x_0 is it possible to find a solution to the above problem, i.e. for what values of x_0 can the constraints be satisfied?

(b) Find the optimal control to (5.1) (for those x_0 you found in (a)).

(c) Let t_f be free, i.e. consider the optimal control problem

$$\min \int_0^{t_f} u(t)dt \quad \text{s.t.} \quad \begin{cases} \dot{x}(t) = -x(t) + u(t), \quad x(0) = x_0 \quad x(t_f) = 0 \\ u \in [0, m]; \ t_f \ge 0 \end{cases}$$

Solve this optimal control problem for the case when $x_0 < 0$.

5.5 Determine the optimal control for the following optimal control problem using PMP

$$\max \int_0^1 (\ln(u) + x)dt \quad \text{subj.to.} \quad \begin{cases} \dot{x} = x - u, \ x(0) = 0 \\ 0 < u \le 1 \end{cases}$$

5.6 The first order optimality conditions (PMP) applied to a continuous time linear quadratic control problem gives rise to the system

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\lambda}(t) \end{bmatrix} = \underbrace{\begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix}}_{\mathcal{H}} \begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix},$$

where the matrix $\mathcal{H}$ is called the *Hamiltonian matrix*. The matrices Q and R are symmetric, i.e. $Q = Q^T$ and $R = R^T$.

(*a*) Show that the matrix $\mathcal{H}$ satisfies the following condition

$$\mathcal{H}^T J + J\mathcal{H} = 0, \quad \text{where} \quad J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}.$$

Matrices that satisfies this condition are called *symplectic*.

(*b*) Compute the eigenvalues for the matrix $\mathcal{H}$ for the scalar case case when $A = a$, $B = b$, $Q = q$ and $R = r$, where q and r are strictly positive numbers and a and b are any real numbers.

(*c*) The eigenvalues are distributed according to a certain symmetry rule. Make a conjecture about the eigenvalues and prove your conjecture.

6. Infinite Time-Horizon Optimal Control

6.1 Consider the following infinite horizon optimal control problem

$$\min \int_0^\infty (x_1(t)^2 + u(t)^2)dt \quad \text{subj. to} \quad \begin{cases} \dot{x}_1(t) = x_2(t), & x_1(0) = x_{10} \\ \dot{x}_2(t) = u(t), & x_2(0) = x_{20} \end{cases} \tag{6.1}$$

(*a*) Formulate the problem on the standard form

$$\min \int_0^\infty (x(t)^T Q x(t) + u(t)^T R u(t))dt \text{ subj. to } \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ x(0) = x_0, \end{cases}$$

i.e. provide the values for all matrices and vectors.

(*b*) Do the factorization $Q = C^T C$ and verify that (C, A) is observable and (A, B) is controllable, i.e. verify that the following matrices

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \qquad \mathcal{C} = \begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix}$$

have full rank (n denotes the dimension of system in the optimal control problem).

(*c*) Determine the optimal stabilizing state feedback solution to the optimal control problem (6.1).

(*d*) Is the solution in (*c*) unique?

(*e*) Verify that the closed loop system is stable?

6.2 Consider the scalar linear quadratic optimal control problem

$$\min \int_0^\infty (3x^2 + u^2)dt \quad \text{subject to} \quad \dot{x} = -x + u, \; x(0) = 1 \tag{6.1}$$

(a) Compute the optimal stabilizing feedback control and the corresponding optimal cost.

(b) Compute the closed loop poles.

Now consider the finite truncation of (6.1)

$$\min \int_0^T (3x^2 + u^2)dt \quad \text{subject to} \quad \dot{x} = -x + u, \ x(0) = 1 \tag{6.2}$$

(c) Use the Hamilton-Jacobi-Bellman equation to compute the optimal feedback control and the corresponding optimal cost.

(d) Let $p(t, T)$ be the Riccati solution corresponding to (6.2), where the final time is made explicit as an argument. Compute $\lim_{T\to\infty} p(t, T)$ and compare with the solution to the ARE corresponding to (6.1).

7. Mixed Problems

7.1 For the following four problems you only need to give a short answer with a brief motivation. Note that if the constraint set is empty, i.e., there does not exist a control satisfying the conditions in the constraint, then the optimal value is ∞.

(a) What is the optimal value for

$$J = \min \int_0^\infty (x^T Q x + u^T R u)dt \quad \text{subj. to} \quad \begin{cases} \dot{x} = Ax + Bu, \\ x(0) = 0 \end{cases}$$

where $Q \geq 0$ and $R > 0$.

(c) What is the optimal value for

$$J = \min t_f \quad \text{subj. to} \quad \begin{cases} \dot{x} = u, \quad x(0) = 1, \ x(t_f) = 0 \\ u \in [0, 1], \ t_f \geq 0 \end{cases}$$

(b) Consider the optimal control problem

$$\min x_2(T) + \int_0^T f_0(x, u)dt \quad \text{subject to} \quad \begin{cases} \dot{x} = f(x, u), \quad x(0) = x_0 \\ x_1(T) = 1 \end{cases}$$

The state vector has n-variables ($x = \begin{bmatrix} x_1 & x_2 & \ldots & x_n \end{bmatrix}^T$). What are the boundary conditions on the adjoint vector λ that can be derived from PMP.

(d) Consider the problem

$$\min \int_0^{t_f} f_0(t, x(t), u(t))dt \quad \text{subj. to} \quad \begin{cases} \dot{x} = f(t, x(t), u(t)) \\ x(0) = x_0, \end{cases}$$

Suppose the solution $(x^*(\cdot), u^*(\cdot))$, and the function $\lambda(\cdot)$ satisfies the conditions of PMP. Assume further that

$$H_{uu}(t, x^*(t), u^*(t), \lambda(t)) = 2$$
$$H_{ux}(t, x^*(t), u^*(t), \lambda(t)) = 1$$
$$H_{xx}(t, x^*(t), u^*(t), \lambda(t)) = 2$$

Is $(x^*(\cdot), u^*(\cdot))$ a local minimum?

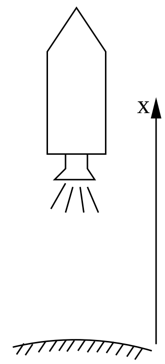

Figure 7.1: Miniumum-time landing of a rocket on the moon.

7.2 Figure 7.1 shows a space craft in the terminal phase of a minimum-time landing on the surface of the moon. The dynamics describing the system are

$$\begin{aligned}
\dot{x}_1(t) &= x_2(t), & x_1(0) &= x_0, \\
\dot{x}_2(t) &= -g - \frac{k}{x_3(t)} u(t), & x_2(0) &= v_0, \\
\dot{x}_3(t) &= u(t), & x_3(0) &= m_0, \\
u(t) &\in [-M, 0]
\end{aligned}$$

where $x_1 = x$, $x_2 = \dot{x}$ and $x_3 = m$, the mass of the rocket. The gravitational constant is denoted g and k is a constant representing the relative exhaust velocity of gases.

(a) Formulate the problem of performing the landing in minimum time as an optimal control problem. The rocket should move from the initial condition $(x_1(t_0), x_2(t_0), x_3(t_0)) = (x_0, v_0, m_0)$ to the surface of the moon $(x_1(t_f), x_2(t_f)) = (0, 0)$ subject to the control constraint $u(t) \in [-M, 0]$.

(b) Use PMP to formulate the two-point boundary-value problem (TPBVP) from which the optimal control can be solved. You do not need to solve the (TPBVP) but all conditions from PMP must be stated.
Remark: You may assume that there are no singular solutions

7.3 PMP gives candidiates for optimality for nonlinear optimal control problems on the form

$$\min \phi(x(t_f)) + \int_0^{t_f} f_0(t, x, u) \quad \text{subject to} \quad \dot{x} = f(t, x, u),\ x(0) = x_0.$$

It is possible to use the second order variation to prove local optimality of such candidiates. The second order optimality conditions reduces to proving that a particular linear quadratic optimal control problem is strictly positive definite. We will touch on this problem for a special case. Consider the following scalar problem

$$
\begin{aligned}
&\min q_0 x(t_f)^2 + \int_0^{t_f} (q(t)x(t)^2 + r(t)u(t)^2)dt \\
&\text{subject to} \begin{cases} \dot{x}(t) = a(t)x(t) + b(t)u(t), \\ x(0) = 0 \end{cases}
\end{aligned}
\tag{7.1}
$$

(a) Prove that

$$
\frac{d}{dt}(p(t)x(t)^2) + q(t)x(t)^2 + r(t)u(t)^2 = r(t)\left(u(t) + \frac{p(t)b(t)}{r(t)}x\right)^2
$$

where $p(t)$ is the solution to the Riccati equation corresponding to (7.1).

(b) Assume $r(t) > 0$. Use (a) to show that

$$
q_0 x(t_f)^2 + \int_0^{t_f} (q(t)x(t)^2 + r(t)u(t)^2)dt > 0
$$

for all nonzero solutions to $\dot{x}(t) = a(t)x(t) + b(t)u(t),\ x(0) = 0$. Note that we don't demand that $q_0 > 0$ or $q > 0$.