Suggested solutions. Exam Maj 23 2013 in SF2852 Optimal Control.

1. **Solution:** We use PMP. The Hamiltonian is

$$H(x,u,\lambda) = u^2 - \lambda x + \lambda u$$

Pointwise minimization gives $u^*(t) = -\lambda(t)/2$. The adjoint equation is

$$\dot{\lambda} = \lambda$$

and there is no boundary constrant specified on λ. This implies $u^*(t) = -\lambda(t)/2 = -e^t\lambda_0$, where λ_0 is a free parameter that must be chosen such that the state constraint is satisfied (this means we have chosen $\lambda(t) = e^t\lambda(0)$ with $\lambda(0) = 2\lambda_0$). We have

$$x(1) = e^{-1} - \int_0^1 e^{-(1-t)}e^t dt\lambda_0 = e^{-1} - \frac{\lambda_0}{2}(e^1 - e^{-1}) = 0$$

This implies

$$\lambda_0 = \frac{1}{e\sinh(1)}.$$

The optimal control is $u^*(t) = -e^t\lambda_0$ and the optimal cost is

$$\int_0^1 (u^*(t))^2 dt = \frac{\lambda_0^2}{2}(e^2 - 1)$$

2. **Solution:**

   (a) The ARE becomes $-2p + 1 = p^2$, which gives $p = -1 \pm \sqrt{2}$. The positive definite solution $p = \sqrt{2} - 1$ corresponds to the stabilizing solution. We get

      i. The optimal stabilizing feedback control $u = (1 - \sqrt{2})x$.

      ii. The optimal cost $J(x(0)) = x(0)^2 p = \sqrt{2} - 1$.

   (b) The closed loop system becomes

$$\dot{x} = -x + (1 - \sqrt{2})x = -\sqrt{2}x$$

   Hence, the closed loop pole is at $s = -\sqrt{2}$.

(c) HJBE gives rise to the Riccati equation

$$\dot{p} - 2p + 1 - p^2 = 0$$

Separation of variables gives

$$\begin{aligned}
&\frac{dp}{(p+1+\sqrt{2})(p+1-\sqrt{2})} = 2\sqrt{2}dt \\
\Leftrightarrow \quad &\ln\left(\frac{p(t)+1-\sqrt{2}}{p(t)+1+\sqrt{2}}\right) = 2\sqrt{2}t + c \\
\Leftrightarrow \quad &\frac{p(t)+1-\sqrt{2}}{p(t)+1+\sqrt{2}} = e^{2\sqrt{2}t+c}
\end{aligned}$$

The boundary condition $p(T) = 0$ gives $c = -2\sqrt{2}T + \ln((1-\sqrt{2})/(1+\sqrt{2}))$. Hence,

$$p(t,T) = \frac{1 - e^{2\sqrt{2}(t-T)}}{1+\sqrt{2}+(\sqrt{2}-1)e^{2\sqrt{2}(t-T)}}$$

The optimal feedback solution is $u(t) = p(t,T)x(t)$ and the optimal cost-to-go is $J(t,x) = p(t,T)x^2$.

(d) We have

$$\lim_{T\to\infty} p(t,T) = \frac{1}{1+\sqrt{2}} = \sqrt{2}-1$$

which is the same as the stabilizing solution to the ARE in problem (a).

3. **Solution:**

(a) Taking the logarithm of the cost we want to maximize

$$\log L(x_1,\ldots,x_T) = -\frac{T-1}{2}\log(2\pi) - \frac{1}{2}\sum_{t=1}^{T-1}(x_t - x_{t+1})^2, \quad (1)$$

or equivalently minimizing

$$\sum_{t=1}^{T-1}(x_t - x_{t+1})^2.$$

This can be written as the dynamic programming problem

$$\min \sum_{t=0}^{T-1} f_0(t, x_t, u_t) \quad \text{subject to} \quad \begin{cases} x_{t+1} = x_t + u_t \\ x_0 = 0 \\ x_t + u_t \in M_{t+1} \\ \text{for } t = 0, 1, \ldots, T-1 \end{cases}$$

where

$$\begin{aligned}\phi(x_T) &= 0\\ f_0(0,0,u_0) &= 0\\ f_0(t,x_t,u_t) &= u_t^2.\end{aligned}$$

The optimal cost is then obtained from (1).

(b) Let

$$J(k,x_k) = \min \sum_{t=k}^{T-1} f_0(t,x_t,u_t) \quad \text{subject to} \quad \begin{cases} x_{t+1} = x_t + u_t \\ x_t + u_t \in M_{t+1} \\ \text{for } t = k, k+1, \ldots, T-1 \end{cases}$$

for $x_k \in M_k$ and $k \in 0, 1, \ldots, T$.

Using dynamic programming, the backward recursion

$$\begin{aligned} J(k,x) &= \min_{u_k + x_k \in M_{k+1}} \{f_0(k,x_k,u_k) + J(k+1, x_k+u_k)\}, \quad k = T-1, T-2, \ldots, 0 \\ J(T,x) &= 0 \end{aligned}$$

gives the optimal solution. From this, we have

$$\begin{aligned}
J(4,x) &= 0, \quad x \text{ for } \in M_4 \\
\\
J(3,4) &= \min_{x \in M_4 = \{2,7,8\}} (x-4)^2 + J(4,x) = 4, \quad (x_4 = 2), \\
J(3,6) &= \min_{x \in M_4 = \{2,7,8\}} (x-6)^2 + J(4,x) = 1, \quad (x_4 = 7), \\
J(3,8) &= \min_{x \in M_4 = \{2,7,8\}} (x-8)^2 + J(4,x) = 0, \quad (x_4 = 8), \\
\\
J(2,2) &= \min_{x \in M_3 = \{4,6,8\}} (x-2)^2 + J(3,x) = 8, \quad (x_3 = 4), \\
J(2,5) &= \min_{x \in M_3 = \{4,6,8\}} (x-5)^2 + J(3,x) = 2, \quad (x_3 = 6), \\
J(2,8) &= \min_{x \in M_3 = \{4,6,8\}} (x-8)^2 + J(3,x) = 0, \quad (x_3 = 8), \\
\\
J(1,2) &= \min_{x \in M_2 = \{2,5,8\}} (x-2)^2 + J(2,x) = 8, \quad (x_2 = 2), \\
J(1,4) &= \min_{x \in M_2 = \{2,5,8\}} (x-4)^2 + J(2,x) = 3, \quad (x_2 = 5), \\
\\
J(0,0) &= \min_{x \in M_1 = \{2,4\}} J(1,x) = 3, \quad (x_1 = 4).
\end{aligned}$$

The optimal path is $(4 \to 5 \to 6 \to 7)$, and the likelihood is

$$L(x_1, \ldots, x_T) = \frac{1}{(2\pi)^{3/2}} \exp\left(-\frac{1}{2} J(0,0)\right) = \frac{e^{-3/2}}{(2\pi)^{3/2}}.$$

4. **Solution:**

The Hamiltonian is given by

$$H(x, u, \lambda) = (u - x)^2 + \lambda(ax + u),$$

and pointwise minimization gives

$$0 = \frac{\partial H}{\partial u} \Longrightarrow u = x - \frac{\lambda}{2}.$$

The Hamiltonian is zero, hence we get

$$0 = H = \lambda\left((1 + a)x - \frac{\lambda}{4}\right)$$

which has the two solutions $\lambda \in \{0, 4(1 + a)x\}$. In (a) the solution $\lambda = 0$ is not stabilizing, and the optimal solution is

$$u = x - 2(1 + a)x \Longrightarrow \dot{x} = -(1 + a)x$$

This gives $x(t) = e^{-(1+a)t}x_0$ and noting that $x - u = 2(1 + a)x$ we have that

$$\int_0^\infty (x - u)^2 dt = 4(1 + a)^2 x_0^2 \int_0^\infty e^{-2(1+a)t} dt = 2(1 + a)x_0^2.$$

In (b), $u = x$, which corresponds to $\lambda = 0$, is stabilizing and the optimal cost is hence 0.

In (c), the solution $u = x$ gives cost zero, but is not stabilizing since then $x \equiv x_0$. The cost can be made arbitrary close to zero by letting $u = (1 - \epsilon)x$ with $\epsilon > 0$ tending to zero.

5. **Solution:**

(a) Letting $y(t) = \int_0^t u(\tau)d\tau$, the optimization becomes

$$\max \int_0^T x(t)dt \quad \text{subject to} \quad \begin{cases} \dot{x} = -x^2 + u \\ \dot{y} = u \\ x(0) = 0 \\ y(0) = 0, \quad y(T) = K, \\ 0 \le u(t) \le 1. \end{cases}$$

The Hamiltonian corresponding to the problem is

$$H(x, u, \lambda) = x + \lambda(-x^2 + u) + \mu u$$

where (λ, μ) are the dual variables. The dynamics for the dual equations are given by

$$\begin{aligned}
\dot{\lambda} &= -\frac{\partial H}{\partial x} = -1 + 2\lambda x \\
\dot{\mu} &= -\frac{\partial H}{\partial y} = 0.
\end{aligned}$$

The boundary condition of the dual system is $\lambda(T) = 0$.

(b) The optimal u is the maximizing argument the Hamiltonian, hence

$$\arg\max_{u\in[0,1]} H((x,y),u,(\lambda,\mu)) = \arg\max_{u\in[0,1]} u(\lambda+\mu) = \begin{cases} 1 & \lambda+\mu > 0 \\ ? & \lambda+\mu = 0 \\ 0 & \lambda+\mu < 0. \end{cases}$$

The control in the second case only affect the dynamics if $\lambda+\mu = 0$ on an interval I. Since $\dot{\mu} = 0$ we must have $0 = \dot{\lambda} = -1 + 2\lambda x$ on I, hence $\lambda x = 1/2$. For this to hold, we must have $\dot{x} = 0$ as well, hence $u = x^2$ on I (note that $\lambda \neq 0$ since $\dot{\lambda} = 0$ and $\lambda = 0$ cannot hold simultaneously). The maximizing control is therefore

$$\arg\max_{u\in[0,1]} H((x,y),u,(\lambda,\mu)) = \arg\max_{u\in[0,1]} u(\lambda+\mu) = \begin{cases} 1 & \lambda+\mu > 0 \\ x^2 & \lambda+\mu = 0 \\ 0 & \lambda+\mu < 0. \end{cases}$$

except possibly at isolated points.

(c) Note first that $0 \leq x(t) < 1$ for any control on $t \in [0,T]$. Due to this, $\dot{\lambda} < 0$ for $\lambda < 1/2$. Therefore $\lambda(T) = 0$ implies that $\lambda(t) > 0$ for all $t \in [0,T)$. We now have $\mu < 0$ since otherwise $\lambda+\mu > 0$ for $t \in [0,T)$ and hence $u(t) = 1$ for $t \in [0,T)$, which is an infeasible control.

In the phase plane (x,λ) there are five regions of interest.

$$\begin{array}{ll}
(1): x\lambda < 1/2, \lambda+\mu \geq 0 & \text{control } u = 1 \\
(2): x\lambda = 1/2, \lambda+\mu = 0 & \text{control } u = x^2 \\
(3): x\lambda \leq 1/2, \lambda+\mu < 0 & \text{control } u = 0 \\
(4): x\lambda \geq 1/2, \lambda+\mu > 0 & \text{invariant (cannot satisfy } \lambda(T) = 0) \\
(5): x\lambda > 1/2, \lambda+\mu \leq 0 & \text{can never be reached.}
\end{array}$$

The only regions that can be part of an optimal trajectory are the first three listed above. At $t = 0$, the state is in (1), since $x(0) = 0$. At time T, the dual state is in (3) since $\lambda(T) = 0$. Note that here $x\lambda \leq 1/2$, hence $\dot{\lambda} \leq 0$. The only possible sequence of states are hence (1) $\to$ (2) $\to$ (3) or (1) $\to$ (3). The possible switching sequences are hence $u = (1, x^2, 0)$ or $(1, 0)$.

(d) Let $K = 1$ and $T > K$. First we want to bound the cost using the control

$$u(t) = \begin{cases} 1 & t < 1 \\ 0 & t \geq 1. \end{cases} \tag{2}$$

First note that $\int_0^1 x dt \leq 1$ since $x(t) \leq 1$. Secondly, for $t \geq 1$, $\dot{x} = -x^2$ implies that

$$x(t) = \frac{1}{t - 1 + \frac{1}{x(1)}}.$$

Integrating, we get

$$\int_1^T x dt = \log((T-1)x(1) + 1) \leq \log(T),$$

hence $\int_0^T x dt \leq 1 + \log(T)$ using this control.
Compare this to the control $u \equiv 1/T$. The ODE $\dot{x} = -x^2 + 1/T$ with $x(0) = 0$ has the solution

$$x(t) = \left(1 - 2\left(1 + e^{2t/\sqrt{T}}\right)^{-1}\right)/\sqrt{T}.$$

When $t \geq \sqrt{T}$ it holds that

$$x(t) \geq \left(1 - \frac{1}{1+e^2}\right)/\sqrt{T} \geq \frac{1}{2\sqrt{T}}$$

hence

$$\int_0^T x dt \geq \int_{\sqrt{T}}^T x dt \geq \frac{T - \sqrt{T}}{2\sqrt{T}} = (\sqrt{T} - 1)/2.$$

Since $\log(T)/\sqrt{T} \to 0$ as $T \to \infty$, it is possible to pick T such that $(\sqrt{T} - 1)/2 > 1 + \log(T)$, (e.g., $T = 1000$). For such T, the control (2) is not optimal, hence the optimal switching sequence for these parameters is $u = (1, x^2, 0)$.