Exam June 2, 2015 in SF2852 Optimal Control.

Examiner: Johan Karlsson, tel. 790 84 40.

Allowed books: The formula sheet and β mathematics handbook.

Solution methods: All conclusions should be properly motivated.

Note! Your personal number must be stated on the cover sheet. Number your pages and write your name on each sheet that you turn in!

Preliminary grades (Credit = exam credit + bonus from homeworks): 23-24 credits give grade Fx (contact examiner asap for further info), 25-27 credits give grade E, 28-32 credits give grade D, 33-38 credits give grade C, 39-44 credits give grade B, and 45 or more credits give grade A.

1. Solve the problem

$$\min \int_0^1 x(t)u(t)dt \quad \text{subject to} \quad \begin{cases} \dot{x} = \left(\frac{1}{2} - u\right)x, \\ x(0) = 1, \quad x(1) = 1, \\ |u(t)| \leq 1. \end{cases}$$

.... (10p)

Hint: Assume that $x > 0$.

2. Determine the optimal stabilizing state feedback control corresponding to the following optimal control problem (cost is not needed explicitly)

$$\min \int_0^\infty \frac{1}{2}(x^2 + u^2)dt \quad \text{subj. to} \quad \dot{x} = x + x^2 + u, \quad x(0) = x_0.$$

.... (10p)

3. Consider the following discrete problem:

$$\min \sum_{k=0}^{\infty} \left(x_k^2 + 2u_k^2\right) \quad \text{subject to} \quad \begin{cases} x_{k+1} = x_k + u_k \\ x_0 = x_i \text{ given.} \end{cases}$$

   (*a*) Determine the optimal feedback and cost when $x \in \mathbf{R}$.
   .... (5p)

   (*b*) Determine the feasible control set that restricts the state x to the intervals $[-\infty, 3] \cup [4, 12]$. Determine the optimal feedback and cost for this problem.
   .... (5p)

4. Background: In speech processing, a sound wave is modeled as a stationary stochastic process over a time frame of 30 ms. It is common to represent such a time frame by its first n cepstral coefficients (a vector in R^n). In this way distances between speech frames are defined as the Euclidean distance between the first n cepstral coefficients.

   One tool for model matching is Dynamic Time Warping (DTW). Consider two sequences $\{x_t\}_{t=0}^N, \{y_t\}_{t=0}^M$, such that $M < N, x_t, y_t \in \mathrm{R}^n$. To determine how "close" the sequences are one can introduce the time scaling τ which is an function $\{0,\dots,N\} \to \{0,\dots,M\}$ and compute

$$\begin{aligned} &\min \sum_{t=0}^{N} \|x_t - y_{\tau(t)}\| \\ &\text{subject to } \ \tau(t) \le \tau(t+1) \le \tau(t)+1, \ \tau(1) = 1, \tau(N) = M. \end{aligned}$$

   The idea is thus to repeat some elements y_t to get a good fit between these two sequences. The time scaling of the sequence y_t makes the data less sensitive to the speed with which the sentences have been pronounced.

   (a) Formulate this minimization as a multistage decision problem (discrete dynamic programming problem).
   .... (6p)

   (b) State the dynamic programming recursion formulas and boundary conditions.
   .... (4p)

5. Consider the following optimal control problem

$$\max x_1(T) \quad \text{subj. to} \quad \begin{cases} \dot{x}_1 = x_2 + u, & x_1(0) = 0 \\ \dot{x}_2 = -x_1, & x_2(0) = 0 \\ \int_0^T u^2(t)dt = c \end{cases}$$

   (a) Reformulate the optimal control problem as a problem on state space form.
   .... (2p)

   (b) Solve the optimal control problem. .... (6p)

   (c) What happens when $T \to \infty$. .... (2p)

Solutions

1. The Hamiltonian function is

$$H(x, u, \lambda) = ux + \lambda(1/2 - u)x$$

Pointwise maximization gives (here we assume $x > 0$)

$$\arg\min H(x, u, \lambda) = \arg\min (1 - \lambda)ux = \begin{cases} 1, & \lambda > 1 \\ -1, & \lambda < 1 \end{cases}$$

In order to determine the number of switches we first determine the adjoint equation

$$\dot{\lambda} = -\frac{\partial}{\partial x} H(x, u, \lambda) = -u - \lambda(1/2 - u) = -\lambda/2 + (\lambda - 1)u$$

Consider the switching function $\sigma = \lambda - 1$. We have

$$\dot{\sigma} = \dot{\lambda} = -\lambda/2 + (\lambda - 1)u$$

which gives

$$\dot{\sigma}_{|\sigma=0} = -1/2$$

Hence, $\sigma(t) = 0$ gives $\dot{\sigma}(t) < 0$, which means that we at most can have one switch. The possible optimal control sequences are $\{-1\}, \{1\}, \{1, -1\}$. The first two must be excluded since they imply that $x(1) < 1$ and $x(1) > 2$, respectively. We must have $u^* \equiv 1$ on $[0, \tau]$ and $u^* \equiv -1$ on $[\tau, T]$. We need to determine the switching time τ. To do this we integrate the system equation

$$\begin{aligned} x(\tau) &= e^{-\frac{1}{2}(1-0)\tau} x(0) = e^{-\frac{1}{2}\tau} \\ x(1) &= e^{\frac{3}{2}(1-\tau)} x(\tau) = e^{\frac{3}{2}(1-\tau)} e^{-\frac{1}{2}\tau} = 1 \end{aligned}$$

which gives

$$\frac{3}{2}(1 - \tau) - \frac{1}{2}\tau = 0 \Rightarrow \tau = 3/4.$$

2. We need to find a positive definite radially unbounded C^1 function that satisfies the HJBE

$$0 = \min_u \{\frac{1}{2}(x^2 + u^2) + V'(x)(x + x^2 + u)\} = \frac{x^2}{2} - \frac{1}{2}(V'(x))^2 + V'(x)(x + x^2)\}$$

The HJBE is an ordinary differential equation in this scalar case. We can easily see that

$$\begin{aligned} V'(x) &= x + x^2 \pm \sqrt{(x + x^2)^2 + x^2} \\ &= x + x^2 + x\sqrt{(1 + x)^2 + 1} \end{aligned}$$

The second equality follows since this case is necessary to obtain a positive definite solution. The optimal feedback control is

$$u^*(x) = -(x + x^2 + x\sqrt{1 + (1 + x)^2})$$

3. (*a*) A function $V(x)$ is equal to the cost function $J^*(x)$ if it is positive definite, quadratically bounded, and satisfies the Bellman equation:

$$\begin{aligned} V(x) &= \min_{u \in U(x)} \{f_0(x, u) + V(f(x, u))\} \\ &= \min_{u \in U(x)} \{x^2 + 2u^2 + V(x + u)\}. \end{aligned}$$

Furthermore, the minimizing argument gives the optimal feedback control. Since the cost function is quadratic and the dynamics are linear, we try a quadratic cost $V(x) = px^2$. This gives

$$\begin{aligned} px^2 &= \min_{u \in U(x)} \{x^2 + 2u^2 + p(x + u)^2\} \\ &= \min_{u \in U(x)} \left\{(2 + p)(u + x/2)^2 + x^2 \left(1 + p - \frac{p^2}{2 + p}\right)\right\}. \end{aligned}$$

The minimum is achieved by $u = -x/2$, and the Bellman equation is thus satisfied if

$$p = 1 + p - \frac{p^2}{2 + p} \Rightarrow p = 1/2 \pm 3/2.$$

The condition of positive definiteness gives $p = 2$. The cost is thus $J^*(x) = 2x^2$ and the optimal feedback is $u = -x/2$.

(*b*) The set of feasible controls are given by $x_k + u_k = x_{k+1} \in [-\infty, 3] \cup [4, 12]$, or equivalently by $u \in U(x) = [-\infty, 3 - x] \cup [4 - x, 12 - x]$.

Note that whenever the sequence $x, x/2, x/2^2, \ldots$ all belong to $[-\infty, 3] \cup [4, 12]$, then the optimal control from (*a*) is feasible and thus the cost is the same as in (*a*). Thus for $x \in [-\infty, 3] \cup [4, 6] \cup [8, 12]$ we have that $J^*(x) = 2x^2$. What remain is to determine $J^*(x)$ and the optimal control for $x \in (6, 8)$. Hence, we would like to compute

$$V(x) = \min_{u \in U(x)} \{x^2 + 2u^2 + V(x + u)\}$$

for $x \in (6, 8)$. However, by noting that $V(x)$ is increasing for $x \geq 0$ we can see that the minimum is always attained for $x + u < 6$, and hence we arrive at

$$\begin{aligned} V(x) &= \min_{x+u\in[-\infty,3]\cup[4,6]} \{x^2+2u^2+2(x+u)^2\} \\ &= \min_{x+u\in[-\infty,3]\cup[4,6]} \left\{4((u+x)-x/2)^2+2x^2\right\} \end{aligned}$$

The minimum (as well as the optimal feedback) is thus $u = 3-x$ for $x \in (6,7]$ and $u = 4-x$ for $x \in [7,8)$, and the minimal cost is

$$V(x) = \begin{cases} 2x^2 & \text{if } x \in [-\infty,3]\cup[4,6]\cup[8,12] \\ (3-x/2)^2+2x^2 & \text{if } x \in (6,7) \\ (4-x/2)^2+2x^2 & \text{if } x \in (7,8). \end{cases}$$

Note that the optimal control is not unique if $x = 7$.

4. (a) The natural way to consider the problem is with τ as the state and $u(t) = \tau(t+1)-\tau(t)$ as control. From the inequalities $\tau(t) \le \tau(t+1) \le \tau(t)+1$ and $\tau(N) = M$, it follows that $M-t \le \tau(N-t) \le M$ for all t. This implies that the admissible controls $U(t,\tau)$ is given by

$$U(t,\tau) = \begin{cases} 0 & \text{for } \tau = M \\ 1 & \text{for } \tau = M-N+t \\ \{0,1\} & \text{for } M-N+t<\tau<M, \end{cases} \tag{1}$$

The control problem now becomes

$$\begin{aligned} &\min \|x_N - y_{\tau(N)}\| + \textstyle\sum_{t=0}^{N-1} \|x_t - y_{\tau(t)}\| \\ &\text{subj. to } \ \tau(t+1) = \tau(t)+u(t),\ \tau(1) = 1, u(t) \in U(t,\tau(t)), \end{aligned}$$

where $U(t,\tau)$ is given by (1).

(b) From the problem statement above it is clear that the dynamic programming recursion is

$$J(t,\tau) = \min_{u\in U(t,\tau)} \left\{\|x_t - y_{\tau(t)}\| + J(t+1,\tau+u)\right\}. \quad t = N-1, N-2, \ldots, 0.$$

with boundary condition

$$J(N,\tau) = \|x_N - y_{\tau(N)}\|.$$

5. (a) If we let $x_3(t) = \int_0^t u^2(s)ds$ then the optimal control problem can be reformulated as

$$\min\ -x_1(T) \quad \text{subj. to} \quad \begin{cases} \dot{x}_1 = x_2 + u, & x_1(0) = 0 \\ \dot{x}_2 = -x_1, & x_2(0) = 0 \\ \dot{x}_3 = u^2, & x_3(0) = 0,\ x_3(T) = c \end{cases}$$

(b) Let us proceed as usual and introduce the Hamiltonian $H(x,u,\lambda) = \lambda_1(x_2+u) - \lambda_2 x_1 + \lambda_3 u^2$. Pointwise minimization gives

$$\arg\min_u H(x,u,\lambda) = \arg\min_u \lambda_1 u + \lambda_3 u^2 = \begin{cases} -\dfrac{\lambda_1}{2\lambda_3}, & \lambda_3 > 0 \\ \infty, & \lambda_3 \le 0 \end{cases}$$

The adjoint system is

$$\begin{cases} \dot{\lambda}_1 = \lambda_2, & \lambda_1(T) = -1 \\ \dot{\lambda}_2 = -\lambda_1, & \lambda_2(T) = 0 \\ \dot{\lambda}_3 = 0, & \lambda_3(T) = ? \end{cases}$$

From the last equation we see that λ_3 must be a constant. It is also clear that $\lambda_3 = k > 0$ since otherwise $u^* = \infty$, which is unreasonable. Solving the adjoint equation gives $\lambda_1(t) = -\cos(T-t)$ and thus

$$u^*(t) = \frac{1}{2k}\cos(T-t)$$

where

$$k = \frac{1}{2\sqrt{c}}\sqrt{\int_0^T \cos(T-t)^2 dt}$$

which follows since $x_3(T) = \int_0^T u^2(t)dt = c$.

(c) $k \to \infty$ as $T \to \infty$, which implies $u^* \to 0$ as $T \to \infty$. For the state we have

$$x_1(T) = \frac{1}{2k}\int_0^T \cos^2(T-t)dt = \sqrt{c}\sqrt{\int_0^T \cos^2(t)dt} \to \infty$$

as $T \to \infty$.