Homework 2: SF2852: Optimal Control Spring 2017

These exercises will be on PMP and application of computational methods for finding numerical solutions to optimal control problems. You will work in groups of at most three persons, and you should hand in a common written report describing your solutions to the questions below. This report should be sent as a pdf file by the time described on the homepage. You should also participate in the seminar and be ready to present your solutions to the questions.

Problem formulation

This project will consider a simple model of a rescue problem. Consider an area in a two-dimensional plane. One person is stuck at a place where an accident has just occurred. He or she has no idea how to get out of the scene, so a rescue worker must guide him or her to the exit. We call the person waiting to be rescued the follower while the rescuer is called the leader. Now assume that the follower can see the leader only when they are close enough to each other. The follower will move towards the leader when he is within sight and the velocity of the follower depends on the distance between them. If the leader is out of sight, the follower will stand still. The velocity of the follower is modelled using the following interaction function

$$g(\beta) = \begin{cases} k\beta, & \text{if } \|\beta\| \le d; \\ \psi(\|\beta\|)\frac{\beta}{\|\beta\|}, & \text{if } d < \|\beta\| \le d + \epsilon; \\ 0, & \text{otherwise,} \end{cases} \tag{1}$$

depicted in Figure 1. Here β is the difference vector between leader and follower, k is a positive constant, and $\psi(\beta)$ is a nonlinear function.

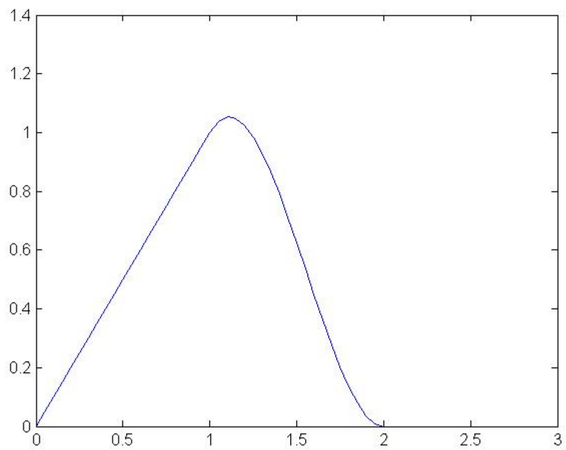

Figure 1: The function $\|\beta\| \to \|g(\beta)\|$

The goal for the leader is to guide the follower as close as possible to the exit in a limited time. At the same time, the leader wants to spend as little energy as possible to finish the task. The overall cost to be minimized is a sum of two cost functions representing the two different aspects. The scene is illustrated in Figure 2. A mathematical model for this problem is

$$
(P) \qquad \begin{aligned}
\min \quad & \|x_f(t_f) - x_{\text{exit}}\|^2 + \int_0^{t_f} \|u(t)\|^2 dt \\
s.t. \quad & \dot{x}_f = g(x_l - x_f), \\
& \dot{x}_l = u, \\
& x_f(t),\ x_l(t),\ u(t) \in \mathbb{R}^2, \\
& x_f(0),\ x_l(0) \text{ given.}
\end{aligned}
$$

Here x_f is the position of the follower, x_l is the position of the leader, x_{exit} is the position of the exit, and $u(t)$ is the control which indicates the velocity of the leader. The nonlinear interaction function g is defined by (1).

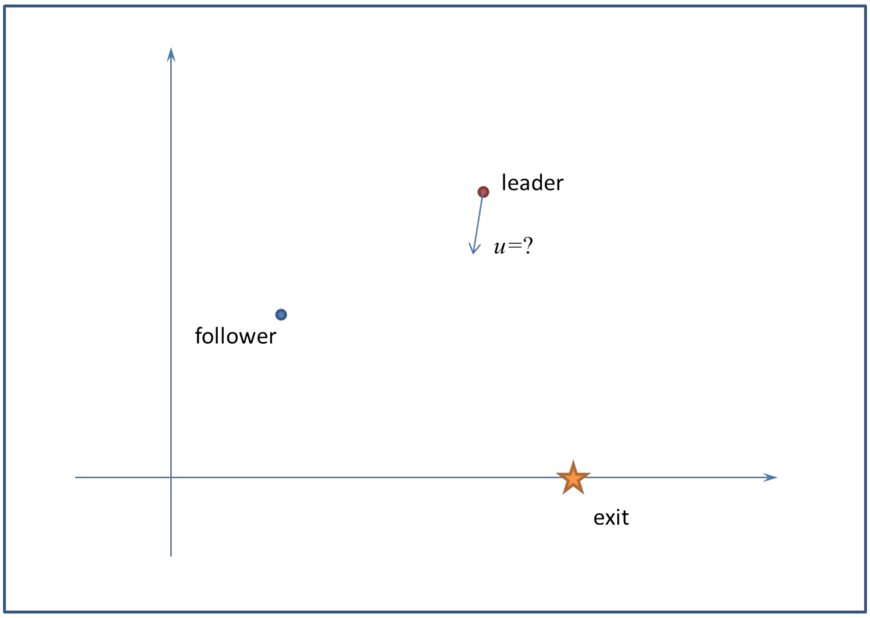

Figure 2: Illustration of the problem (P)

Now assume that the time and the positions are given by

$$
t_f = 4, \quad x_f(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad x_l(0) = \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \quad x_{\text{exit}} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}.
$$

Furthermore, let the function g be specified by the parameters

$$
k = 1, \quad d = 1, \quad \epsilon = 1, \quad \psi(s) = 3s^3 - 14s^2 + 20s - 8.
$$

(Note that $\psi(1) = 1$, $\psi'(1) = 1$, $\psi(2) = 0$, and $\psi'(2) = 0$, hence g is a smooth function.)

Problem 1

Write down the Hamiltonian, the pointwise minimization and the Two Point Boundary Value Problem (TPBVP) in the PMP algorithm for the optimal control problem (P). You are *not* required to solve the TPBVP.

Problem 2

Solve the optimal control problem (P) using the following two methods.

(a) **Discretization** (page 122 in the lecture notes). Discretize the time interval and solve the minimization problem using the MATLAB function *fmincon*. Try different step-sizes and initial values for the solver.

(b) **Consistent approximations** (page 128 in the lecture notes). Choose an orthogonal basis for the approximation of the control function $u(t)$ and solve the optimization problem with respect to the coefficients of the basis. You can for example choose

$$\varphi_1 = \frac{1}{2},\ \varphi_2 = \frac{1}{\sqrt{2}} \sin\left(\frac{\pi t}{2}\right),\ \varphi_3 = \frac{1}{\sqrt{2}} \cos\left(\frac{\pi t}{2}\right),$$
$$\varphi_4 = \frac{1}{\sqrt{2}} \sin\left(\frac{2\pi t}{2}\right),\ \varphi_5 = \frac{1}{\sqrt{2}} \cos\left(\frac{2\pi t}{2}\right),\ \ldots$$

as the orthogonal basis. Try different initial values for the solver. (HINT: the corresponding optimization problem will be unconstrained)

Problem 3

Change the initial position of the leader to $x_l(0) = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$, and solve the problem again (using one of the methods). Compare the results you obtained. Did something unexpected occur? If so, explain it. Note that non-convex optimization problems may have several local optima. Can you find at least two local optima to this problem?

Problem 4

Use your creativity to extend the problem so that it will be more realistic. How does this extension affect the cost function, the dynamics, the optimal control, and the initial and final state constraints. Prepare to discuss the extended problem, from aspects of how well it represents the basic rescue problem, and how it can be solved.

Solve at least one variation of (P) and present the solution. Compare the optimal solution to the one for (P) and explain differences in the respective solutions.