EXCERPT FROM 6th EDITION OF HILLIER & LIEBERMAN

Continuous Review Model with Fixed Delivery Lag and Backlogging: In Sec. 17.3, a deterministic continuous review model, i.e., the economic lot-size model, was considered. The demand was assumed to be at a known constant rate. This model was classified as continuous review in that the inventory was continuously monitored and orders were placed at any time; i.e., orders were placed whenever the inventory level dropped to the *order point* (sometimes called the *trigger point*). This ordering procedure is in contrast to the models considered so far in this section, which assume stochastic demand and periodic review; i.e., the inventory is monitored only at the beginning of each period, and orders are placed only at these times. The model considered now is analogous to the economic lot-size model (continuous review), but where the demand for the item is *stochastic* and there is a *fixed delivery lead time* before an order is received. Only (s, S)-type policies are considered; i.e., when the inventory level falls to a level s, an order is placed to bring the inventory level up to S (a quantity $Q = S - s$ is ordered). This model is often called a *lot-size reorder-point model;* a quantity Q is ordered whenever the inventory level reaches the reorder level s. Unsatisfied demand is assumed to be filled immediately upon replenishment of the inventory; i.e., unsatisfied demand is backlogged.

To describe the model further, we first need to summarize three different ways of measuring the amount of inventory.

1. The **inventory on hand** is the number of units physically located in inventory. Thus, this quantity must be *nonnegative.*
2. The **inventory level** is the inventory on hand *minus* the amount of (backlogged) unsatisfied demand. Unsatisfied demand can occur (temporarily) only after the inventory on hand has dropped to zero, so unsatisfied demand causes the inventory level to be *negative.*
3. The **inventory position** is the inventory level *plus* the amount ordered but not yet received. The inventory position normally will be kept *nonnegative.*

The model can be described in detail as follows. Inventory is stockpiled and used as demand dictates. When the inventory position reaches s, an order is placed for Q units to bring the inventory position up to level S. There is a fixed delivery lead time (often called *lag time*) of length λ before the order is received. The demand for units from inventory during time λ is assumed to be a continuous random variable D having a probability density function denoted by $\varphi_D(\xi)$ and mean

$$E(D) = a\lambda,$$

where a is the expected number of items demanded per unit time.

Figure 17.10 illustrates how both the inventory level (the solid curve) and the inventory position vary over time. Note that this diagram can be viewed as a series of cycles, with a *cycle* defined as the time between receipt of consecutive orders. The figure includes a cycle where the demand during period λ is relatively large, which eventually causes the inventory level to go negative (where this unsatisfied demand is backlogged to be met when the order arrives). The inventory position differs from the inventory level only during the period of a delivery lag time, so the inventory position during these periods is shown by the dashed curve.

The costs to be considered are

K = setup cost for placing an order,

c = unit cost for each unit purchased,

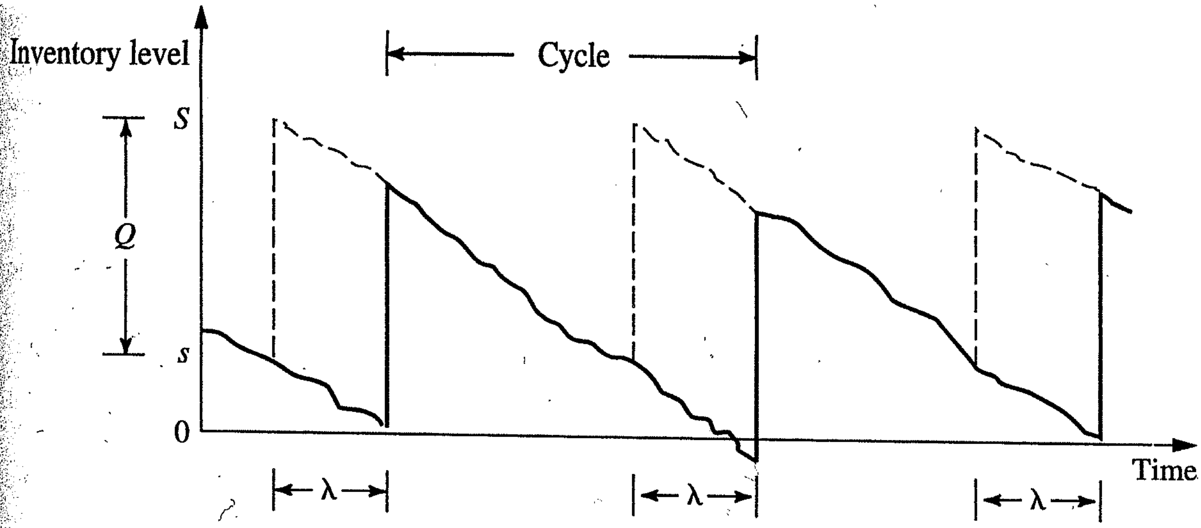

Figure 17.10 Diagram of the inventory level (the solid curve) as a function of time for the stochastic continuous review model. When the inventory position differs from the inventory level, it is shown by a dashed curve.

h = holding cost per unit of inventory on hand per unit time.

p = shortage cost per unit short (until next order arrives), independent of duration of shortage.

The inventory policy is to track the inventory position so that when the inventory position reaches s, an order of size Q is placed; this order will be delivered after a period of length λ. The problem is to determine when to place an order (find the order point s) and what size it should be (find the order quantity Q), in order to minimize the expected total cost per unit time, $C(Q, s)$

$$C(Q, s) = E(\text{OC}) + E(\text{HC}) + E(\text{SC}),$$

where $E(\text{OC})$ = expected ordering cost per unit time,
$E(\text{HC})$ = expected holding cost per unit time,
$E(\text{SC})$ = expected shortage cost per unit time.

To evaluate these terms, we assume initially that λ is sufficiently small that there is never more than a single order outstanding and that the reorder point s (based on the inventory position) is always nonnegative. The first assumption guarantees that the inventory on hand when an order is received will always fall above the reorder point, because otherwise more than one order would be outstanding. If p/h is sufficiently large, as is usually the case in practice, these assumptions are generally satisfied.

The expected ordering cost per unit time $E(\text{OC})$ is simply the ordering cost incurred per cycle *times* the expected number of cycles per unit of time:

$$\text{Ordering cost per cycle} = K + cQ.$$

Because a is the expected demand per unit time,

$$\text{Expected number of cycles per unit time} = \frac{a}{Q}.$$

Therefore,

$$E(\text{OC}) = \frac{a}{Q}(K + cQ).$$

The expected holding cost per unit time is

$$E(\text{HC}) = h\,E(\text{average inventory on hand during a cycle}).$$

The expected value of the average inventory level during a cycle can be obtained by averaging the expected inventory on hand at the beginning and end of a cycle. From Fig. 17.10, the expected inventory on hand at the beginning of the cycle is given by $S - a\lambda$, and the expected inventory on hand at the end of the cycle is approximately $s - a\lambda$. (The latter quantity is approximate because it ignores the possibility of a *negative* inventory level, which leaves *zero* inventory on hand, at the end of the cycle.) Hence, the expected average amount of inventory on hand during a cycle can be approximated by

$$\frac{(S - a\lambda) + (s - a\lambda)}{2} = \frac{Q + s - a\lambda + s - a\lambda}{2} = \frac{Q}{2} + s - a\lambda,$$

so that

$$E(\text{HC}) = h\left(\frac{Q}{2} + s - a\lambda\right).$$

The expected shortage cost per unit time $E(\text{SC})$ is the expected shortage cost incurred per cycle *times* the expected number of cycles per unit time (already obtained as a/Q). Since a shortage can occur only when the demand during the delivery lag time exceeds s, the expected shortage cost per cycle is

$$p\int_s^\infty (\xi - s)\varphi_D(\xi)\, d\xi,$$

so that

$$E(\text{SC}) = \frac{a}{Q}p\int_s^\infty (\xi - s)\varphi_D(\xi)\, d\xi.$$

Adding the expressions for $E(\text{OC})$, $E(\text{HC})$, and $E(\text{SC})$ leads to

$$C(Q, s) = \frac{aK}{Q} + ac + h\left(\frac{Q}{2} + s - a\lambda\right) + \frac{pa}{Q}\int_s^\infty (\xi - s)\varphi_D(\xi)\, d\xi.$$

Because there are two decision variables (Q and s), the optimal values (Q^* and s^*) are found by setting the corresponding partial derivatives equal to zero, i.e.,

$$\frac{\partial C(Q, s)}{\partial Q} = \frac{-aK}{Q^2} + \frac{h}{2} - \frac{pa\int_s^\infty (\xi - s)\varphi_D(\xi)\, d\xi}{Q^2} = 0$$

$$\frac{\partial C(Q, s)}{\partial s} = h - \frac{pa\int_s^\infty \varphi_D(\xi)\, d\xi}{Q} = 0.$$

Solving these equations simultaneously[1] leads to

(1) $$Q^* = \sqrt{\frac{2a\left[K + p\int_{s^*}^\infty (\xi - s^*)\varphi_D(\xi)\, d\xi\right]}{h}},$$

(2) $$\int_{s^*}^\infty \varphi_D(\xi)\, d\xi = \frac{hQ^*}{pa}.$$

[1] Note that the optimal values of Q and s are independent of c, the unit cost of the units ordered. The total number of units ordered is independent of the values of Q and s, so c can be neglected in determining the optimal values of these parameters.

Unfortunately, solving these equations simultaneously and obtaining a general closed-form expression for Q^* and s^* are not possible, but the following iterative procedure will lead to close approximations of these quantities.

1. As an initial step, assume that p equals zero and obtain a value of Q from Eq. (1). (Note that this equation with $p = 0$ is just the expression for Q in the deterministic economic lot-size model when shortages are not permitted.)
2. Solve for s in Eq. (2), using the value of Q found in step 1.
3. Using the value of s found in step 2, solve for a new Q, using Eq. (1).
4. Repeat steps 2 and 3 until successive values of Q and of s are sufficiently close.

In practice, this procedure will generally converge in just a few iterations.

Remarks: Several remarks can be made about this continuous review model.

1. Note that in Eq. (2), the integral $\int_{s^*}^{\infty} \varphi_D(\xi)\, d\xi$ is just the probability that the random variable demand D during the lead time exceeds s^*, that is, $P\{D > s^*\}$. Hence, $hQ^*/(pa)$ must fall between 0 and 1. If the algorithm ever leads to a value of $hQ/(pa) > 1$, this is an indication that the shortage cost (relative to the holding cost) is too small, with the result that the shortage by the end of a cycle will tend to be large. This would contradict the approximation of neglecting stockouts that was made in the calculation of the expected holding cost per unit time, so that the derived formulas would become inappropriate.

2. If the lead time is close to the average cycle length Q/a, more than one order may be outstanding. Using the inventory position as the measure of inventory still leads to an operational rule, i.e., order when the inventory position reaches s. However, the number of stockouts may become too large to be neglected in the expected holding cost calculations.

3. The quantity $(s - a\lambda)$ is known as the *safety stock,* and it represents "protection" against a stockout during the delivery lag time. The probability of a stockout is $P\{D > s\} = \int_{s}^{\infty} \varphi_D(\xi)\, d\xi$. By Eq. (2), this probability also equals $hQ^*/(pa)$ when s^* and Q^* are used.

4. Because there is no closed-form solution of Eqs. (1) and (2), it is worth considering some special cases for the probability distribution of demand. Consider the *uniform distribution* over the range from 0 to t, so that the probability density function is

$$\varphi_D(\xi) = \begin{cases} \dfrac{1}{t}, & \text{if } 0 \le \xi \le t, \\ 0 & \text{otherwise.} \end{cases}$$

Then Eq. (2) leads to

$$\int_{s}^{\infty} \varphi_D(\xi)\, d\xi = 1 - \frac{s}{t},$$

so that

$$s^* = \frac{t(pa - hQ^*)}{pa} = t\left(1 - \frac{hQ^*}{pa}\right).$$

Furthermore, from Eq. (1),

$$\int_s^\infty (\xi - s)\varphi_D(\xi)\, d\xi = \frac{t}{2} + \frac{s^2}{2t} - s$$

and

$$Q^* = \sqrt{\frac{2aK + apt + aps^{*2}/t - 2aps^*}{h}}.$$

By substituting the expression for s^* into the right-hand side of this last equation and then squaring both sides, the equation reduces algebraically to

$$Q^* = \sqrt{\frac{ap}{ap - ht}}\sqrt{\frac{2aK}{h}}.$$

Notice the close similarity to the economic lot-size formula ($Q^* = \sqrt{2aK/h}$) found at the beginning of Sec. 17.3 for the *deterministic* continuous review model with no shortages permitted.

5. Now consider the case where the demand during the delivery lag time λ has an exponential distribution with mean $a\lambda$, so that

$$\varphi_D(\xi) = \left(\frac{1}{a\lambda}\right)e^{-\xi/(a\lambda)}, \qquad \text{for } \xi \geq 0.$$

Since

$$\int_s^\infty \varphi_D(\xi)\, d\xi = e^{-s/(a\lambda)},$$

Eq. (2) yields

$$s^* = -a\lambda \ln \frac{hQ^*}{pa}.$$

Furthermore, from Eq. (1),

$$\int_s^\infty (\xi - s)\varphi_D(\xi)\, d\xi = a\lambda e^{-s/(a\lambda)},$$

and

$$Q^* = \sqrt{\frac{2a}{h}\left(K + a\lambda p e^{-s^*/(a\lambda)}\right)}.$$

Proceeding just as for the preceding case of a uniform distribution of demand, we now substitute the expression for s^* into this expression for Q^*, square both sides of this latter expression, and reduce it algebraically to obtain

$$Q^* = a\lambda + \sqrt{a^2\lambda^2 + \frac{2aK}{h}}.$$

Once again, note the similarity to the economic lot-size formula.

When the demand has either a uniform or an exponential distribution, a routine is available in your OR Courseware for obtaining s^* and Q^*.

EXAMPLE: Consider the speaker example presented in Sec. 17.1. It was assumed that $K = 12{,}000$ and $h = 0.30$ per speaker per month, with a fixed demand rate of $a = 8{,}000$ per month and with instantaneous production of the speakers each time a production run is scheduled.

Now suppose that there actually is a *lag time* of $\lambda = 1$ month between ordering a production run to produce speakers and having the speakers ready for assembly into television sets. Also suppose that interruptions in the production of television sets cause the demand for speakers during this lag time to be a random variable, but still with a mean of $a = 8{,}000$ per month. The shortage cost is $p = \$5$ per speaker not ready as soon as it is needed.

If demand has a uniform distribution over the range from 0 to $t = 16{,}000$, the corresponding formulas give

$$Q^* = \sqrt{\frac{8{,}000(5)}{8{,}000(5) - 0.3(16{,}000)}}\sqrt{\frac{2(8{,}000)(12{,}000)}{0.3}} = 26{,}968,$$

$$s^* = 16{,}000\left[1 - \frac{0.3(26{,}968)}{5(8{,}000)}\right] = 12{,}764.$$

The probability of a stockout during a cycle then is

$$P\{D > s^*\} = \frac{hQ^*}{pa} = 0.20.$$

If demand instead has an exponential distribution with a mean of 8,000, the corresponding formulas give

$$Q^* = 8{,}000 + \sqrt{8{,}000^2 + \frac{2(8{,}000)(12{,}000)}{0.3}} = 34{,}533,$$

$$s^* = -8{,}000 \ln\frac{0.3(34{,}533)}{5(8{,}000)} = 10{,}807.$$

The probability of a stockout during a cycle then is given by

$$P\{D > s^*\} = \frac{hQ^*}{pa} = 0.26.$$

CONTINUOUS REVIEW MODEL WITH FIXED DELIVERY LAG AND NO BACKLOGGING: This model is identical to the preceding model except that unsatisfied demand now is assumed to be lost, i.e., unsatisfied demand will not be backlogged. Therefore, lost revenue now is included in the shortage cost.

The derivation of the costs contains the same approximations that were made in the backlogging case, so the subsequent expressions will lead to approximate results. The expressions for the expected ordering cost per unit of time $E(\text{OC})$ and the expected shortage cost per unit of time $E(\text{SC})$ are the same for both models. The only cost that differs is the expected holding cost per unit of time $E(\text{HC})$.

Recall that the backlogging model approximated the average inventory on hand during a cycle as the average of the inventory level at the beginning of the cycle and at the end of the cycle. Without backlogging, the inventory level cannot go negative, so the inventory level at the beginning of the cycle now will be larger by the number of units short (if any) at the end of the preceding cycle. Similarly, the inventory level at the end of a cycle now will be larger than for the backlogging model by the number of units short (if any) at that time. Consequently, the current model needs to adjust the expression for *expected average* inventory level for the backlogging model by adding

$$\text{Expected number of units short at the end of a cycle} = \int_s^\infty (\xi - s)\varphi_D(\xi)\,d\xi.$$

This adjustment yields

$$E(\text{HC}) = h\left[\frac{Q}{2} + s - a\lambda + \int_s^\infty (\xi - s)\varphi_D(\xi)\,d\xi\right].$$

Therefore, the expected total cost per unit time $C(Q, s)$ is

$$C(Q, s) = \frac{aK}{Q} + ac + h\left[\frac{Q}{2} + s - a\lambda + \int_s^\infty (\xi - s)\varphi_D(\xi)\,d\xi\right] + \frac{pa}{Q}\int_s^\infty (\xi - s)\varphi_D(\xi)\,d\xi.$$

Because there are two decision variables (Q and s), the optimal values (Q^* and s^*) are found by setting the corresponding partial derivatives equal to zero, i.e.,

$$\frac{\partial C(Q, s)}{\partial Q} = \frac{-aK}{Q^2} + \frac{h}{2} - \frac{pa\int_s^\infty (\xi - s)\varphi_D(\xi)\,d\xi}{Q^2} = 0$$

$$\frac{\partial C(Q, s)}{\partial s} = h\int_0^s \varphi_D(\xi)\,d\xi - \frac{pa\int_s^\infty \varphi_D(\xi)\,d\xi}{Q} = 0.$$

Solving these equations simultaneously[1] leads to

(3) $$Q^* = \sqrt{\frac{2a\left[K + p\int_{s^*}^\infty (\xi - s^*)\varphi_D(\xi)\,d\xi\right]}{h}}$$

(4) $$\int_{s^*}^\infty \varphi_D(\xi)\,d\xi = \frac{hQ^*}{hQ^* + pa}.$$

Unfortunately, solving these equations simultaneously to obtain a general closed-form expression for Q^* and s^* is not possible. However, the same iterative procedure as given for the backlogging model [but now using Eqs. (3) and (4) in place of Eqs. (1) and (2)] can be used to closely approximate Q^* and s^*. In practice, this procedure will generally converge in just a few iterations.

When the demand has either a uniform or an exponential distribution, a routine is available in your OR Courseware for obtaining s^* and Q^*.