Markov chains in discrete time

Consider a stochastic process in discrete time.

Think of it as a sequence $X_0, X_1, X_2, \ldots$ denoted $\{X_t\}$ of stochastic (random) variables that takes values in $1, 2, 3, \ldots, M$. (M could be ∞ finite/infinite state space)

X_t usually represents the state of a system at time t.

Examples:

- X_t = wind condition $\{1 = \text{calm}, 2 = \text{breeze}, 3 = \text{storm}\}$ at a particular place on day $t = 0, 1, 2, \ldots$
- X_t = number of items in stock of a particular item on day $t = 0, 1, 2, \ldots$
- X_t = accumulated sum of points after t rolls of a die.
- X_t = number of rabbits living on Gärdet at time $t = 0, 1, 2, \ldots$
- X_t = number of complaint phone calls to the help desk day $t = 0, 1, 2 \ldots$
- X_t - condition of patient $\{1 = \text{stable}, 2 = \text{manic}, 3 = \text{depressive}\}$ on day t.

Def: A stochastic process $\{X_t\}$ is said to have the <u>Markovian property</u> if

$$P(X_{t+1} = j \mid X_0 = k_0, X_1 = k_1, \ldots, X_t = k_t) = P(X_{t+1} = j \mid X_t = k_t)$$

for $t = 0, 1, \ldots$ and every sequence $j, k_0, k_1, \ldots, k_t$

The conditional probability of a future event depends only on the present state and <u>not</u> on all past states.

The present state holds all the information about the process to predict the future.

Def: A stochastic process $\{X_t\}$ $(t = 0, 1, \ldots)$ is a <u>Markov chain</u> if it has the Markovian property.

Which of the examples above are Markov chains?

Def. A stochastic process $\{X_t\}$ has <u>stationary transition probabilities</u>

if $P(X_{t+1}=j \mid X_t=i) = P(X_1=j \mid X_0=i)$ for all $t=1,2,\dots$

Which of the examples above have stationary trans. prob. ?

We will consider Markov chains with stationary transition probabilities.

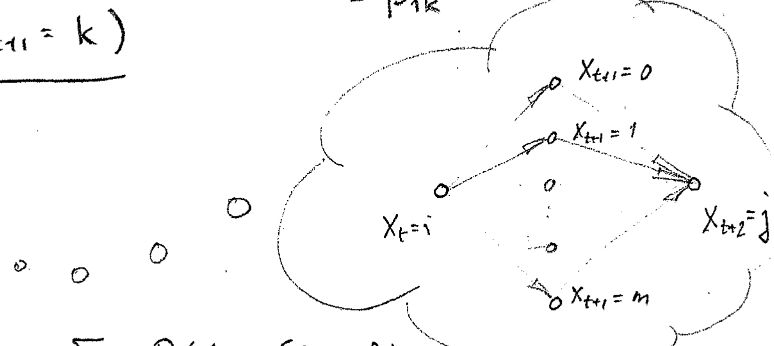
Define $p_{ij} = P(X_{t+1}=j \mid X_t=i)$ (independent of t)

Many of the properties of the process are determined by p_{ij}.

Now $p_{ij}^{(2)} \triangleq P(X_{t+2}=j \mid X_t=i) = \sum_{k=0}^{m} P(X_{t+2}=j, X_{t+1}=k \mid X_t=i)$

$$= \sum_{k=0}^{m} \underbrace{P(X_{t+2}=j \mid X_{t+1}=k, X_t=i)}_{= \underbrace{P(X_{t+2}=j \mid X_{t+1}=k)}_{=p_{kj}}} \underbrace{P(X_{t+1}=k \mid X_t=i)}_{=p_{ik}}$$

$$= \sum_{k=0}^{m} p_{kj}\, p_{ik}$$



where we have used that $P(A) = \sum_{k\in B} P(A \cap \{B=k\})$

law of total probability

and $P(A\cap\{B=k\}) = P(A \mid \{B=k\}) \cdot P(\{B=k\})$

Define the (one-step) transition matrix $P = [p_{ij}] = \begin{bmatrix} p_{00} & p_{01} & \cdots & p_{0m} \\ p_{10} & p_{11} & & \\ \vdots & & & \\ p_{m0} & & & p_{mm} \end{bmatrix}$

and the (two-step) transition matrix $P^{(2)} = [p_{ij}^{(2)}] = \begin{bmatrix} p_{00}^{(2)} & p_{01}^{(2)} & & p_{0m}^{(2)} \\ p_{10}^{(2)} & p_{11}^{(2)} & & \\ & & & \\ p_{m0}^{(2)} & & & p_{mm}^{(2)} \end{bmatrix}$

Then we have just shown that $P^{(2)} = P^2$.